HL Paper 3

A relation S is defined on $\mathbb{R}$ by aSb if and only if $ab > 0$.

A relation R is defined on a non-empty set A. R is symmetric and transitive but not reflexive.

a. Show that S is [4]

(i) not reflexive;

(ii) symmetric;

(iii) transitive.

b. Explain why there exists an element $a \in A$ that is not related to itself. [1]

c. Hence prove that there is at least one element of A that is not related to any other element of A. [6]

Markscheme

a. (i) $0S0$ is not true so S is not reflexive ***A1AG***

(ii) $aSb \Rightarrow ab > 0 \Rightarrow ba > 0 \Rightarrow bSa$ so S is symmetric ***R1AG***

(iii) aSb and $bSc \Rightarrow ab > 0$ and $bc > 0 \Rightarrow ab^2c > 0 \Rightarrow ac > 0$ ***M1***

since $b^2 > 0$ (as b could not be 0) $\Rightarrow aSc$ so S is transitive ***R1AG***

Note: *R1* is for indicating that $b^2 > 0$.

[4 marks]

b. since R is not reflexive there is at least one element a belonging to A such that a is not related to a ***R1AG***

[1 mark]

c. argue by contradiction: suppose that a is related to some other element b, *ie*, aRb ***M1***

since R is symmetric aRb implies bRa ***R1A1***

since R is transitive aRb and bRa implies aRa ***R1A1***

giving the required contradiction ***R1***

hence there is at least one element of A that is not related to any other member of A ***AG***

[6 marks]

Examiners report

a. [N/A]
[N/A]

b. [N/A]

Let $f : G \to H$ be a homomorphism between groups $\{G,\ *\}$ and $\{H,\ \circ\}$ with identities e_G and e_H respectively.

a. Prove that $f(e_G) = e_H$. [2]

b. Prove that $\mathrm{Ker}(f)$ is a subgroup of $\{G,\ *\}$. [6]

Markscheme

a. let $a \in G$ and $f(a) \in H$

f is a homomorphism so $f(a * e_G) = f(a) \circ f(e_G)$ ***(M1)***

$f(a) = f(a) \circ f(e_G)$ ***A1***

$e_H = f(e_G)$ ***AG***

[2 marks]

b. from part (a) $e_G \in \mathrm{Ker}(f)$ and associativity follows from G ***R1***

let $a,\ b \in \mathrm{Ker}(f)$

$f(a * b) = f(a) \circ f(b) = e_H \circ e_H = e_H$ ***A1***

hence closed since $a * b \in \mathrm{Ker}(f)$

$e_H = f(a^{-1} * a) = f(a^{-1}) \circ f(a) = f(a^{-1}) \circ e_H = f(a^{-1})$ ***M1A1***

hence $a^{-1} \in \mathrm{Ker}(f)$ ***R1***

hence $\mathrm{Ker}(f)$ is subgroup of G ***AG***

[6 marks]

Examiners report

a. [N/A]

b. [N/A]

A, B and C are three subsets of a universal set.

Consider the sets $P = \{1,\ 2,\ 3\}$, $Q = \{2,\ 3,\ 4\}$ and $R = \{1,\ 3,\ 5\}$.

a.i. Represent the following set on a Venn diagram, [1]

$A \Delta B$, the symmetric difference of the sets A and B;

a.ii. Represent the following set on a Venn diagram, [1]

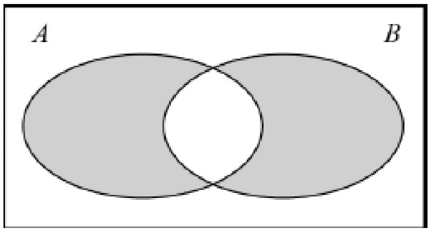
$A \cap (B \cup C)$.

b.i. For sets P, Q and R, verify that $P \cup (Q \Delta R) \neq (P \cup Q) \Delta (P \cup R)$. [4]

b.ii. In the context of the distributive law, describe what the result in part (b)(i) illustrates. [2]

Markscheme

a.i.

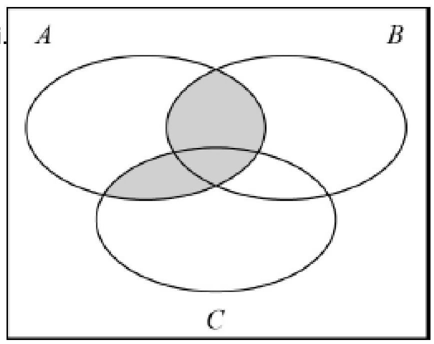


A1

[1 mark]

***Note:** Accept alternative set configurations*

a.ii.



A1

Note: Accept alternative set configurations.

[1 mark]

b.i. LHS:

$Q \Delta R = \{1,\ 2,\ 4,\ 5\}$ ***(A1)***

$P \cup (Q \Delta R) = \{1,\ 2,\ 3,\ 4,\ 5\}$ ***A1***

RHS:

$P \cup Q = \{1,\ 2,\ 3,\ 4\}$ and $P \cup R = \{1,\ 2,\ 3,\ 5\}$ ***(A1)***

$(P \cup Q) \Delta (P \cup R) = \{4,\ 5\}$ ***A1***

hence $P \cup (Q \Delta R) \neq (P \cup Q) \Delta (P \cup R)$ ***AG***

[4 marks]

b.ii. the result shows that union is not distributive over symmetric difference ***A1R1***

Notes: Award ***A1*** for "union is not distributive" and ***R1*** for "over symmetric difference". Condone use of $\cup$ and Δ.

[2 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]

The function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n + (-1)^n$.

a. Prove that $f \circ f$ is the identity function. [6]

b.i. Show that f is injective. [2]

b.ii. Show that f is surjective. [1]

Markscheme

a. **METHOD 1**

$(f \circ f)(n) = n + (-1)^n + (-1)^{n+(-1)^n}$ ***M1A1***

$= n + (-1)^n + (-1)^n \times (-1)^{(-1)^n}$ ***(A1)***

considering $(-1)^n$ for even and odd n ***M1***

if n is odd, $(-1)^n = -1$ and if n is even, $(-1)^n = 1$ and so $(-1)^{\pm 1} = -1$ ***A1***

$= n + (-1)^n - (-1)^n$ ***A1***

= n and so $f \circ f$ is the identity function ***AG***

METHOD 2

$(f \circ f)(n) = n + (-1)^n + (-1)^{n+(-1)^n}$ ***M1A1***

$= n + (-1)^n + (-1)^n \times (-1)^{(-1)^n}$ ***(A1)***

$= n + (-1)^n \times \left(1 + (-1)^{(-1)^n}\right)$ ***M1***

$(-1)^{\pm 1} = -1$ ***R1***

$1 + (-1)^{(-1)^n} = 0$ ***A1***

$(f \circ f)(n) = n$ and so $f \circ f$ is the identity function ***AG***

METHOD 3

$(f \circ f)(n) = f\left(n + (-1)^n\right)$ ***M1***

considering even and odd n ***M1***

if n is even, $f(n) = n + 1$ which is odd ***A1***

so $(f \circ f)(n) = f(n+1) = (n+1) - 1 = n$ ***A1***

if n is odd, $f(n) = n - 1$ which is even ***A1***

so $(f \circ f)(n) = f(n-1) = (n-1) + 1 = n$ ***A1***

$(f \circ f)(n) = n$ in both cases

hence $f \circ f$ is the identity function ***AG***

[6 marks]

b.i. suppose $f(n) = f(m)$ ***M1***

applying f to both sides $\Rightarrow n = m$ ***R1***

hence f is injective ***AG***

[2 marks]

b.ii. $m = f(n)$ has solution $n = f(m)$ ***R1***

hence surjective ***AG***

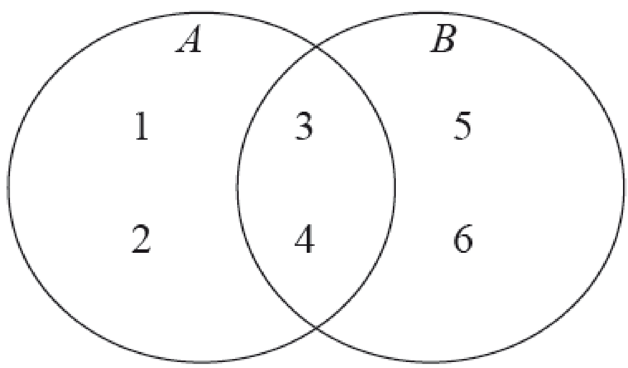
[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

Let $\{G,\ \circ\}$ be the group of all permutations of $1,\ 2,\ 3,\ 4,\ 5,\ 6$ under the operation of composition of permutations.

Consider the following Venn diagram, where $A = \{1,\ 2,\ 3,\ 4\},\ B = \{3,\ 4,\ 5,\ 6\}$.



a. (i) Write the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 1 & 5 \end{pmatrix}$ as a composition of disjoint cycles. [3]

(ii) State the order of α.

b. (i) Write the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ as a composition of disjoint cycles. [2]

(ii) State the order of β.

c. Write the permutation $\alpha \circ \beta$ as a composition of disjoint cycles. [2]

d. Write the permutation $\beta \circ \alpha$ as a composition of disjoint cycles. [2]

e. State the order of $\{G,\ \circ\}$. [2]

f. Find the number of permutations in $\{G,\ \circ\}$ which will result in A, B and $A \cap B$ remaining unchanged. [2]

Markscheme

a. (i) $(1\;3\;6\;5)(2\;4)$ ***A1A1***

(ii) 4 ***A1***

Note: In (b) (c) and (d) single cycles can be omitted.

[3 marks]

b. (i) $(1\;6\;2\;4\;5)(3)$ ***A1***

(ii) 5 ***A1***

[2 marks]

c. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 3 & 4 \end{pmatrix} = (1\;5\;3\;6\;4)(2)$ ***(M1)A1***

[2 marks]

d. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 6 & 1 \end{pmatrix} = (1\;3\;2\;5\;6)(4)$ ***(M1)A1***

Note: Award ***A2A0*** for (c) and (d) combined, if answers are the wrong way round.

[2 marks]

e. $6! = 720$ ***A2***

[2 marks]

f. any composition of the cycles (1 2), (3 4) and (5 6) ***(M1)***

so $2^3 = 8$ ***A1***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]

The binary operations $\odot$ and $*$ are defined on $\mathbb{R}^+$ by

$$a \odot b = \sqrt{ab} \text{ and } a * b = a^2 b^2.$$

Determine whether or not

a. $\odot$ is commutative; [2]

b. $*$ is associative; [4]

c. $*$ is distributive over $\odot$; [4]

d. $\odot$ has an identity element. [3]

Markscheme

a. $a \odot b = \sqrt{ab} = \sqrt{ba} = b \odot a$ ***A1***

since $a \odot b = b \odot a$ it follows that $\odot$ is commutative ***R1***

[2 marks]

b. $a * (b * c) = a * b^2c^2 = a^2b^4c^4$ ***M1A1***

$(a * b) * c = a^2b^2 * c = a^4b^4c^2$ ***A1***

these are different, therefore $*$ is not associative ***R1***

Note: Accept numerical counter-example.

[4 marks]

c. $a * (b \odot c) = a * \sqrt{bc} = a^2bc$ ***M1A1***

$(a * b) \odot (a * c) = a^2b^2 \odot a^2c^2 = a^2bc$ ***A1***

these are equal so $*$ is distributive over $\odot$ ***R1***

[4 marks]

d. the identity e would have to satisfy

$a \odot e = a$ for all a ***M1***

now $a \odot e = \sqrt{ae} = a \Rightarrow e = a$ ***A1***

therefore there is no identity element ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Let $\{G,\ *\}$ be a finite group that contains an element a (that is not the identity element) and $H = \{a^n | n \in \mathbb{Z}^+\}$, where

$a^2 = a * a,\ a^3 = a * a * a$ etc.

Show that $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$.

Markscheme

since G is closed, H will be a subset of G

closure: $p,\ q \in H \Rightarrow p = a^r,\ q = a^s,\ r,\ s \in \mathbb{Z}^+$ ***A1***

$p * q = a^r * a^s = a^{r+s}$ ***A1***

$r + s \in \mathbb{Z}^+ \Rightarrow p * q \in H$ hence H is closed ***R1***

associativity follows since $*$ is associative on G ***(R1)***

EITHER

identity: let the order of a in G be $m \in \mathbb{Z}^+,\ m \geqslant 2$ ***M1***

then $a^m = e \in H$ ***R1***

inverses: $a^{m-1} * a = e \Rightarrow a^{m-1}$ is the inverse of a ***A1***

$(a^{m-1})^n * a^n = e$, showing that a^n has an inverse in H ***R1***

hence H is a subgroup of G ***AG***

OR

since $(G,\ *)$ is a finite group, and H is a non-empty closed subset of G, then $(H,\ *)$ is

a subgroup of $(G,\ *)$ ***R4***

Note: To receive the ***R4***, the candidate must explicitly state the theorem, *i.e.* the three given conditions, and conclusion.

[8 marks]

Examiners report

This question was generally answered very poorly, if attempted at all. Candidates failed to realize that the property of closure needed to be properly proved. Others used negative indices when the question specifically states that the indices are positive integers.

The set A contains all positive integers less than 20 that are congruent to 3 modulo 4.

The set B contains all the prime numbers less than 20.

The set C is defined as $C = \{7,\ 9,\ 13,\ 19\}$.

a.i. Write down all the elements of A and all the elements of B. [2]

a.ii. Determine the symmetric difference, $A \Delta B$, of the sets A and B. [2]

b.i. Write down all the elements of $A \cap B,\ A \cap C$ and $B \cup C$. [3]

b.ii. Hence by considering $A \cap (B \cup C)$, verify that in this case the operation $\cap$ is distributive over the operation $\cup$. [3]

Markscheme

a.i. the elements of A are: 3, 7, 11, 15, 19 ***A1***

the elements of B are 2, 3, 5, 7, 11, 13, 17, 19 ***A1***

Note: Accept $A = \{3,\ 7,\ 11,\ 15,\ 19\}$ and $B = \{2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19\}$

[2 marks]

a.ii. attempt to determine $A \backslash B \cup B \backslash A$ or $(A \cup B) \cap (A \cap B)'$ ***(M1)***

symmetric difference $= \{2,\ 5,\ 13,\ 15,\ 17\}$ ***A1***

Note: Allow ***(M1)A1FT***.

[2 marks]

b.i. the elements of $A \cap B$ are 3, 7, 11 and 19 ***A1***

the elements of $A \cap C$ are 7 and19 ***A1***

the elements of $B \cup C$ are 2, 3, 5, 7, 9, 11, 13, 17 and 19 ***A1***

Note: Accept $A \cap B = \{3,\ 7,\ 11,\ 19\},\ A \cap C = \{7,\ 19\}$ and $B \cup C = \{2,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 17,\ 19\}$.

[3 marks]

b.ii. we need to show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ***(M1)***

$A \cap (B \cup C) = \{3,\ 7,\ 11,\ 19\}$ ***A1***

$(A \cap B) \cup (A \cap C) = \{3,\ 7,\ 11,\ 19\}$ ***A1***

hence showing the required result

Note: Allow ***(M1)A1FTA1FT***.

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]

The relation *R* is defined on $\mathbb{R} \times \mathbb{R}$ such that $(x_1,\ y_1)R(x_2,\ y_2)$ if and only if $x_1y_1 = x_2y_2$.

a. Show that *R* is an equivalence relation. [5]

b. Determine the equivalence class of *R* containing the element $(1,\ 2)$ and illustrate this graphically. [4]

Markscheme

a. *R* is an equivalence relation if

R is reflexive, symmetric and transitive ***A1***

$x_1y_1 = x_1y_1 \Rightarrow (x_1,\ y_1)R(x_1,\ y_1)$ ***A1***

so *R* is reflexive

$(x_1,\ y_1)R(x_2,\ y_2) \Rightarrow x_1y_1 = x_2y_2 \Rightarrow x_2y_2 = x_1y_1 \Rightarrow (x_2,\ y_2)R(x_1,\ y_1)$ ***A1***

so *R* is symmetric

$(x_1,\ y_1)R(x_2,\ y_2)$ and $(x_2,\ y_2)R(x_3,\ y_3) \Rightarrow x_1y_1 = x_2y_2$ and $x_2y_2 = x_3y_3$ ***M1***

$\Rightarrow x_1y_1 = x_3y_3 \Rightarrow (x_1,\ y_1)R(x_3,\ y_3)$ ***A1***

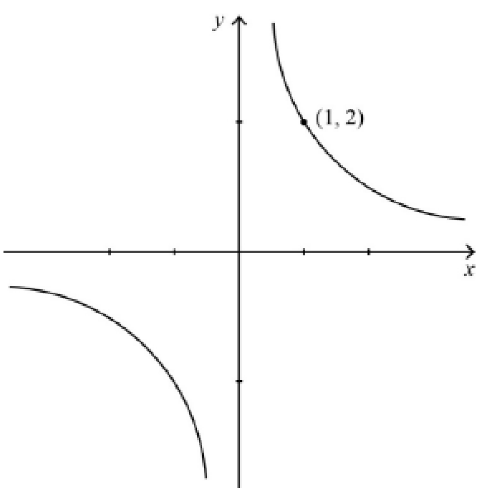
so *R* is transitive

R is an equivalence relation ***AG***

[5 marks]

b. $(x,\ y)R(1,\ 2)$ ***(M1)***

the equivalence class is $\{(x,\ y)|xy = 2\}$ ***A1***



correct graph ***A1***

$(1,\ 2)$ indicated on the graph ***A1***

Note: Award last ***A1*** only if plotted on a curve representing the class.

[4 marks]

Examiners report

a. [N/A]
b. [N/A]

The group $\{G, \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where $\times_7$ denotes multiplication modulo 7.

a. (i) Write down the Cayley table for $\{G, \times_7\}$. [10]

(ii) Determine whether or not $\{G, \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H.

b. The group $\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and $\circ$ denotes composition of permutations. [6]

(i) Show that $\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not $\{G, \times_7\}$ and $\{K, \circ\}$ are isomorphic.

Markscheme

a. (i) the Cayley table is

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

A3

Note: Deduct 1 mark for each error up to a maximum of 3.

(ii) by considering powers of elements, ***(M1)***

it follows that 3 (or 5) is of order 6 ***A1***

so the group is cyclic ***A1***

(iii) we see that 2 and 4 are of order 3 so the subgroup of order 3 is $\{1, 2, 4\}$ ***M1A1***

(iv) the element of order 2 is 6 ***A1***

the coset is $\{3, 5, 6\}$ ***A1***

[10 marks]

b. (i) consider for example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \textit{M1A1}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \textit{M1A1}$$

Note: Award ***M1A1M1A0*** if both compositions are done in the wrong order.

Note: Award ***M1A1M0A0*** if the two compositions give the same result, if no further attempt is made to find two permutations which are not commutative.

these are different so the group is not Abelian ***R1AG***

(ii) they are not isomorphic because $\{G,\ \times_7\}$ is Abelian and $\{K,\ \circ\}$ is not ***R1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

The set of all permutations of the elements $1,\ 2,\ \ldots 10$ is denoted by H and the binary operation $\circ$ represents the composition of permutations.

The permutation $p = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$ generates the subgroup $\{G,\ \circ\}$ of the group $\{H,\ \circ\}$.

a. Find the order of $\{G,\ \circ\}$. [2]

b. State the identity element in $\{G,\ \circ\}$. [1]

c. Find [4]

(i) $p \circ p$;

(ii) the inverse of $p \circ p$.

d. (i) Find the maximum possible order of an element in $\{H,\ \circ\}$. [3]

(ii) Give an example of an element with this order.

Markscheme

a. the order of $(G,\ \circ)$ is $\mathrm{lcm}(6,\ 4)$ ***(M1)***

$= 12$ ***A1***

[2 marks]

b. $(1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (8)\ (9)\ (10)$ ***A1***

Note: Accept ( ) or a word description.

[1 mark]

c. (i) $p \circ p = (1\ 3\ 5)(2\ 4\ 6)(7\ 9)(810)$ ***(M1)A1***

(ii) its inverse $= (1\ 5\ 3)(2\ 6\ 4)(7\ 9)(810)$ ***A1A1***

Note: Award ***A1*** for cycles of 2, ***A1*** for cycles of 3.

[4 marks]

d. (i) considering LCM of length of cycles with length $2,\ 3$ and 5 ***(M1)***

30 ***A1***

(ii) *eg* $(1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$ ***A1***

Note: allow FT as long as the length of cycles adds to 10 and their LCM is consistent with answer to part (i).

Note: Accept alternative notation for each part

[3 marks]

Total [10 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The relation R is defined on the set $\mathbb{N}$ such that for $a\ ,\ b \in \mathbb{N}\ ,\ aRb$ if and only if $a^3 \equiv b^3(\bmod 7)$.

a. Show that R is an equivalence relation. [6]

b. Find the equivalence class containing 0. [2]

c. Denote the equivalence class containing n by C_n . [3]

List the first six elements of C_1.

d. Denote the equivalence class containing n by C_n . [3]

Prove that $C_n = C_{n+7}$ for all $n \in \mathbb{N}$.

Markscheme

a. *reflexive*: $a^3 - a^3 = 0\ ,\ \Rightarrow R$ is reflexive ***R1***

symmetric: if $a^3 \equiv b^3(\bmod 7)$, then $b^3 \equiv a^3(\bmod 7)$ ***M1***
$\Rightarrow R$ is symmetric ***R1***
transitive: $a^3 = b^3 + 7n$ and $b^3 = c^3 + 7m$ ***M1***
then $a^3 = c^3 + 7(n+m)$
$\Rightarrow a^3 \equiv c^3(\bmod 7)$ ***R1***
$\Rightarrow R$ is transitive ***A1***
and is an equivalence relation ***AG***
Note: Allow arguments that use $a^3 - b^3 \equiv 0(\bmod 7)$ *etc.*

[6 marks]

b. $\{0,\ 7,\ 14,\ 21,\ \dots\}$ ***A2***

[2 marks]

c. $\{1,\ 2,\ 4,\ 8,\ 9,\ 11\}$ ***A3***

Note: Deduct 1 mark for each error or omission.

[3 marks]

d. consider $(n+7)^3 = n^3 + 21n^2 + 147n + 343 = n^3 + 7N$ ***M1A1***

$\Rightarrow n^3 \equiv (n+7)^3(\bmod 7) \Rightarrow n$ and $n+7$ are in the same equivalence class ***R1***

[3 marks]

Examiners report

a. Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

b. Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

c. Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

d. Candidates were mostly aware of the conditions required to show an equivalence relation although many seemed unsure as to the degree of detail required to show that the different conditions are met for the example in this question. In part (b) many candidates found the correct set although a number were unable to write down the set correctly, including or excluding elements that were not part of the equivalence class. Part (c) saw candidate being less successful than (b) and relatively few candidates were able to prove the equivalence class in part (d) although there were a number of very good solutions.

The function f is defined by $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ where $f(x,\ y) = \left(\sqrt{xy},\ \frac{x}{y}\right)$

a. Prove that f is an injection. [5]

b. (i) Prove that f is a surjection. [8]

(ii) Hence, or otherwise, write down the inverse function f^{-1}.

Markscheme

a. let $(a,\ b)$ and $(c,\ d) \in \mathbb{R}^+ \times \mathbb{R}^+$

suppose that $f(a,\ b) = f(c,\ d)$ ***(M1)*** so that $\sqrt{ab}=\sqrt{cd}$ and $\frac{a}{b}=\frac{c}{d}$ ***A1***

leading to either $a^2=c^2$ or $b^2=d^2$ or equivalent ***M1***

state $a=c$ and $b=d$ ***A1***

this shows that f is an injection since $f(a,\ b)=f(c,\ d)\Rightarrow(a,\ b)=(c,\ d)$ ***R1AG***

Note: Accept final statement seen anywhere for ***R1***.

[5 marks]

b. (i) now let $(u,\ v)\in\mathbb{R}^+\times\mathbb{R}^+$ and suppose that $f(x,\ y)=(u,\ v)$ ***(M1)***

then, $u=\sqrt{xy},\ v=\frac{x}{y}$ ***A1***

attempt to eliminate x or y ***M1***

$\Rightarrow x=uv^{1/2};\ y=uv^{-1/2}$ ***A1A1***

this shows that f is a surjection since, given $(u,\ v)$, there exists $(x,\ y)$ such that $f(x,\ y)=(u,\ v)$ ***R1AG***

Note: Accept final statement, seen anywhere, for ***R1***.

(ii) $f^{-1}(x,\ y)=(xy^{1/2},\ xy^{-1/2})$ ***A1A1***

[8 marks]

Examiners report

a. Those candidates who formulated their responses in terms of the basic mathematical definitions of injectivity and surjectivity were usually successful. Otherwise, verbal attempts such as 'f is one-to-one $\Rightarrow$ f is injective' or 'g is surjective because its range equals its codomain', received no credit.

b. (i) Those candidates who formulated their responses in terms of the basic mathematical definitions of injectivity and surjectivity were usually successful. Otherwise, verbal attempts such as 'f is one-to-one $\Rightarrow$ f is injective' or 'g is surjective because its range equals its codomain', received no credit.

(ii) It was surprising to see that some candidates were unable to relate what they had done in part (b)(i) to this part.

The relation $R=$ is defined on $\mathbb{Z}^+$ such that aRb if and only if $b^n-a^n\equiv 0(\bmod p)$ where $n,\ p$ are fixed positive integers greater than 1.

a. Show that R is an equivalence relation. [7]

b. Given that $n=2$ and $p=7$, determine the first four members of each of the four equivalence classes of R. [5]

Markscheme

a. since $a^n-a^n=0,$ ***A1***

it follows that (aRa) and R is reflexive ***R1***

if aRb so that $b^n-a^n\equiv 0(\bmod p)$ ***M1***

then, $a^n-b^n\equiv 0(\bmod p)$ so that bRa and R is symmetric ***A1*** if aRb and bRc so that $b^n - a^n \equiv 0(\bmod p)$ and $c^n - b^n \equiv 0(\bmod p)$ ***M1***

adding, (it follows that $c^n - a^n \equiv 0(\bmod p)$) ***M1***

so that aRc and R is transitive ***A1***

Note: Only accept the correct use of the terms "reflexive, symmetric, transitive".

[7 marks]

b. we are now given that aRb if $b^2 - a^2 \equiv 0(\bmod 7)$

attempt to find at least one equivalence class ***(M1)***

the equivalence classes are

$\{1, 6, 8, 13, \ldots\}$ ***A1***

$\{2, 5, 9, 12, \ldots\}$ ***A1***

$\{3, 4, 10, 11, \ldots\}$ ***A1***

$\{7, 14, 21, 28, \ldots\}$ ***A1***

[5 marks]

Examiners report

a. Most candidates were familiar with the terminology of the required conditions to be satisfied for a relation to be an equivalence relation. The execution of the proofs was variable. It was grating to see such statements as R is symmetric because $aRb = bRa$ or $aRa = a^n - a^n = 0$, often without mention of $\bmod p$, and such responses were not fully rewarded.

b. This was not well answered. Few candidates displayed a strategy to find the equivalence classes.

Let c be a positive, real constant. Let G be the set $\{x \in \mathbb{R} | -c < x < c\}$. The binary operation $*$ is defined on the set G by $x * y = \frac{x+y}{1+\frac{xy}{c^2}}$.

a. Simplify $\frac{c}{2} * \frac{3c}{4}$. [2]

b. State the identity element for G under $*$. [1]

c. For $x \in G$ find an expression for x^{-1} (the inverse of x under $*$). [1]

d. Show that the binary operation $*$ is commutative on G . [2]

e. Show that the binary operation $*$ is associative on G . [4]

f. (i) If $x,\ y \in G$ explain why $(c-x)(c-y) > 0$. [2]

(ii) Hence show that $x + y < c + \frac{xy}{c}$.

g. Show that G is closed under $*$. [2]

h. Explain why $\{G, *\}$ is an Abelian group. [2]

Markscheme

a. $\frac{c}{2} * \frac{3c}{4} = \frac{\frac{c}{2}+\frac{3c}{4}}{1+\frac{1}{2}\cdot\frac{3}{4}}$ ***M1***

$= \frac{\frac{5c}{4}}{\frac{11}{8}} = \frac{10c}{11}$ ***A1***

[2 marks]

b. identity is 0 ***A1***

[1 mark]

c. inverse is $-x$ ***A1***

[1 mark]

d.

$x * y = \frac{x+y}{1+\frac{xy}{c^2}},\ y * x = \frac{y+x}{1+\frac{yx}{c^2}}$ ***M1***

(since ordinary addition and multiplication are commutative)

$x * y = y * x$ so $*$ is commutative ***R1***

Note: Accept arguments using symmetry.

[2 marks]

e.

$(x * y) * z = \frac{x+y}{1+\frac{xy}{c^2}} * z = \frac{\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)+z}{1+\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)\frac{z}{c^2}}$ ***M1***

$= \frac{\frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}}{\frac{\left(1+\frac{xy}{c^2}+\frac{xz}{c^2}+\frac{yz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}} = \frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)}$ ***A1***

$$x * (y * z) = x * \left(\frac{y+z}{1+\frac{yz}{c^2}}\right) = \frac{x+\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}{1+\frac{x}{c^2}\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}$$

$$= \frac{\frac{\left(x+\frac{xyz}{c^2}+y+z\right)}{\left(1+\frac{yz}{c^2}\right)}}{\frac{\left(1+\frac{yz}{c^2}+\frac{xy}{c^2}+\frac{xz}{c^2}\right)}{\left(1+\frac{yz}{c^2}\right)}} = \frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)} \quad \textbf{\textit{A1}}$$

since both expressions are the same $*$ is associative ***R1***

Note: After the initial ***M1A1***, correct arguments using symmetry also gain full marks.

[4 marks]

f. (i) $c > x$ and $c > y \Rightarrow c - x > 0$ and $c - y > 0 \Rightarrow (c-x)(c-y) > 0$ ***R1AG***

(ii) $c^2 - cx - cy + xy > 0 \Rightarrow c^2 + xy > cx + cy \Rightarrow c + \frac{xy}{c} > x + y$ (as $c > 0$)

so $x + y < c + \frac{xy}{c}$ ***M1AG***

[2 marks]

g. if $x,\ y \in G$ then $-c - \frac{xy}{c} < x + y < c + \frac{xy}{c}$

thus $-c\left(1 + \frac{xy}{c^2}\right) < x + y < c\left(1 + \frac{xy}{c^2}\right)$ and $-c < \frac{x+y}{1+\frac{xy}{c^2}} < c$ ***M1***

(as $1 + \frac{xy}{c^2} > 0$) so $-c < x * y < c$ ***A1***

proving that G is closed under $*$ ***AG***

[2 marks]

h. as $\{G, *\}$ is closed, is associative, has an identity and all elements have an inverse ***R1***

it is a group ***AG***

as $*$ is commutative ***R1***

it is an Abelian group ***AG***

[2 marks]

Examiners report

a. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

b. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

c. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

d. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

e. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

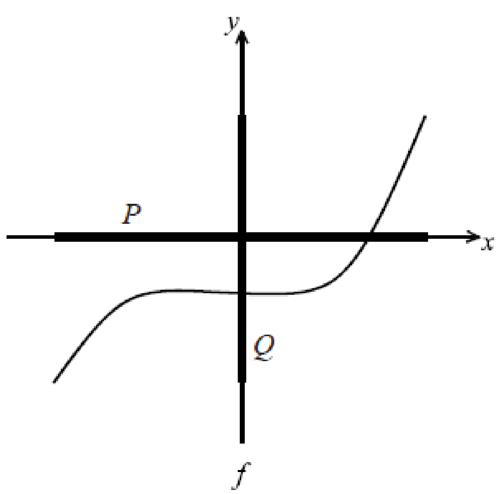
f. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term "state" in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

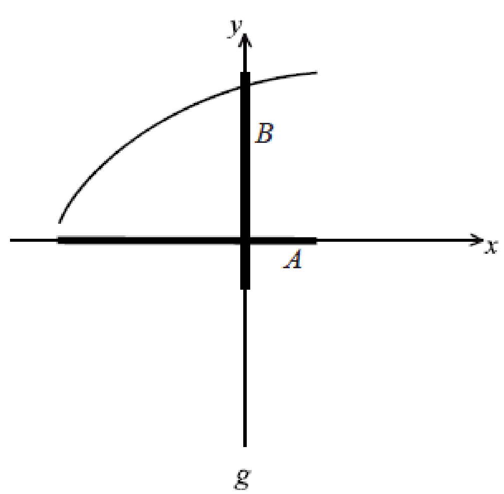
g. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

h. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

---

a. Below are the graphs of the two functions $F : P \to Q$ and $g : A \to B$. [4]





Determine, with reference to features of the graphs, whether the functions are injective and/or surjective.

b. Given two functions $h : X \to Y$ and $k : Y \to Z$. [9]

Show that

(i) if both h and k are injective then so is the composite function $k \circ h$;

(ii) if both h and k are surjective then so is the composite function $k \circ h$.

Markscheme

a. f is surjective because every horizontal line through Q meets the graph somewhere ***R1***

f is not injective because it is a many-to-one function ***R1***

g is injective because it always has a positive gradient ***R1***

(accept horizontal line test reasoning)

g is not surjective because a horizontal line through the negative part of B would not meet the graph at all ***R1***

[4 marks]

b. (i) **EITHER**

Let $x_1,\ x_2 \in X$ and $y_1 = h(x_1)$ and $y_2 = h(x_2)$ ***M1***

Then

$k \circ (h(x_1)) = k \circ (h(x_2))$

$\Rightarrow k(y_1) = k(y_2)$ ***A1***

$\Rightarrow y_1 = y_2$ (k is injective) ***A1***

$\Rightarrow h(x_1) = h(x_2)$ ($h(x_1) = y_1$ and $h(x_2) = y_2$) ***A1***

$\Rightarrow x_1 \equiv x_2$ (h is injective) ***A1***

Hence $k \circ h$ is injective ***AG***

OR

$\mathrm{x}_1,\ x_2 \in X,\ x_1 \neq x_2$ ***M1***

since h is an injection $\Rightarrow h(x_1) \neq h(x_2)$ ***A1***

$h(x_1),\ h(x_2) \in Y$ ***A1***

since k is an injection $\Rightarrow k\left(h(x_1)\right) \neq k\left(h(x_2)\right)$ ***A1***

$k\left(h(x_1)\right),\ k\left(h(x_2)\right) \in \mathbb{Z}$ ***A1***

so $k \circ h$ is an injection. ***AG***

(ii) h and k are surjections and let $z \in \mathbb{Z}$

Since k is surjective there exists $y \in Y$ such that $k(y) = z$ ***R1***

Since h is surjective there exists $x \in X$ such that $h(x) = y$ ***R1***

Therefore there exists $x \in X$ such that

$k \circ h(x) = k\left(h(x)\right)$

$= k(y)$ ***R1***

$= z$ ***A1***

So $k \circ h$ is surjective ***AG***

[9 marks]

Examiners report

a. ‘Using features of the graph’ should have been a fairly open hint but too many candidates contented themselves with describing what injective and surjective meant rather than explaining which graph had which properties. Candidates found considerable difficulty with presenting a convincing argument in part (b).

b. ‘Using features of the graph’ should have been a fairly open hint but too many candidates contented themselves with describing what injective and surjective meant rather than explaining which graph had which properties. Candidates found considerable difficulty with presenting a convincing argument in part (b).

Consider the group $\{G,\ \times_{18}\}$ defined on the set $\{1,\ 5,\ 7,\ 11,\ 13,\ 17\}$ where $\times_{18}$ denotes multiplication modulo 18. The group $\{G,\ \times_{18}\}$ is shown in the following Cayley table.

$\times_{18}$	**1**	**5**	**7**	**11**	**13**	**17**
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

The subgroup of $\{G,\ \times_{18}\}$ of order two is denoted by $\{K,\ \times_{18}\}$.

a.i. Find the order of elements 5, 7 and 17 in $\{G,\ \times_{18}\}$. [4]

a.ii. State whether or not $\{G,\ \times_{18}\}$ is cyclic, justifying your answer. [2]

b. Write down the elements in set K. [1]

c. Find the left cosets of K in $\{G,\ \times_{18}\}$. [4]

Markscheme

a.i. considering powers of elements ***(M1)***

5 has order 6 ***A1***

7 has order 3 ***A1***

17 has order 2 ***A1***

[4 marks]

a.ii. G is cyclic ***A1***

because there is an element (are elements) of order 6 ***R1***

Note: Accept “there is a generator”; allow ***A1R0***.

[3 marks]

b. $\{1,\ 17\}$ ***A1***

[1 mark]

c. multiplying $\{1,\ 17\}$ by each element of G ***(M1)***

$\{1,\ 17\},\ \{5,\ 13\},\ \{7,\ 11\}$ ***A1A1A1***

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]

A group $\{D,\ \times_3\}$ is defined so that $D = \{1,\ 2\}$ and $\times_3$ is multiplication modulo 3.

A function $f : \mathbb{Z} \to D$ is defined as $f : x \mapsto \begin{cases} 1, & x \text{ is even} \\ 2, & x \text{ is odd} \end{cases}$.

a. Prove that the function f is a homomorphism from the group $\{\mathbb{Z},\ +\}$ to $\{D,\ \times_3\}$. [6]

b. Find the kernel of f. [3]

c. Prove that $\{\text{Ker}(f),\ +\}$ is a subgroup of $\{\mathbb{Z},\ +\}$. [4]

Markscheme

a. consider the cases, a and b both even, *one* is even and *one* is odd and a and b are both odd ***(M1)***

calculating $f(a+b)$ and $f(a) \times_3 f(b)$ in at least one case ***M1***

if a is even and b is even, then $a+b$ is even

so $f(a+b) = 1.\quad f(a) \times_3 f(b) = 1 \times_3 1 = 1$ ***A1***

so $f(a+b) = f(a) \times_3 f(b)$

if *one* is even and *the other* is odd, then $a+b$ is odd

so $f(a+b) = 2.\quad f(a) \times_3 f(b) = 1 \times_3 2 = 2$ ***A1***

so $f(a+b) = f(a) \times_3 f(b)$

if a is odd and b is odd, then $a+b$ is even

so $f(a+b) = 1.\quad f(a) \times_3 f(b) = 2 \times_3 2 = 1$ ***A1***

so $f(a+b) = f(a) \times_3 f(b)$

as $f(a+b) = f(a) \times_3 f(b)$ in all cases, so $f : \mathbb{Z} \to D$ is a homomorphism ***R1AG***

[6 marks]

b. 1 is the identity of $\{D,\ \times_3\}$ ***(M1)(A1)***

so $\text{Ker}(f)$ is all even numbers ***A1***

[3 marks]

c. **METHOD 1**

sum of any two even numbers is even so closure applies ***A1***

associative as it is a subset of $\{\mathbb{Z}, +\}$ ***A1***

identity is 0, which is in the kernel ***A1***

the inverse of any even number is also even ***A1***

METHOD 2

$\ker(f) \neq \emptyset$

$b^{-1} \in \ker(f)$ for any b

$ab^{-1} \in \ker(f)$ for any a and b

Note: Allow a general proof that the Kernel is always a subgroup.

[4 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

---

a. Associativity and commutativity are two of the five conditions for a set S with the binary operation $*$ to be an Abelian group; state the other three conditions. [2]

b. The Cayley table for the binary operation $\odot$ defined on the set $T = \{p, q, r, s, t\}$ is given below. [15]

$\odot$	p	q	r	s	t
p	s	r	t	p	q
q	t	s	p	q	r
r	q	t	s	r	p
s	p	q	r	s	t
t	r	p	q	t	s

(i) Show that exactly three of the conditions for $\{T, \odot\}$ to be an Abelian group are satisfied, but that neither associativity nor commutativity are satisfied.
(ii) Find the proper subsets of T that are groups of order 2, and comment on your result in the context of Lagrange's theorem.
(iii) Find the solutions of the equation $(p \odot x) \odot x = x \odot p$.

Markscheme

a. closure, identity, inverse ***A2***

Note: Award ***A1*** for two correct properties, ***A0*** otherwise.

[2 marks]

b. (i) closure: there are no extra elements in the table ***R1***

identity: s is a (left and right) identity ***R1***
inverses: all elements are self-inverse ***R1***

commutative: no, because the table is not symmetrical about the leading diagonal, or by counterexample ***R1***
associativity: for example, $(pq)t = rt = p$ ***M1A1***
not associative because $p(qt) = pr = t \neq p$ ***R1***
Note: Award ***M1A1*** for 1 complete example whether or not it shows non-associativity.

(ii) $\{s,\, p\},\ \{s,\, q\},\ \{s,\, r\},\ \{s,\, t\}$ ***A2***
Note: Award ***A1*** for 2 or 3 correct sets.

as 2 does not divide 5, Lagrange's theorem would have been contradicted if *T* had been a group ***R1***

(iii) any attempt at trying values ***(M1)***
the solutions are q, r, s and t ***A1A1A1A1***
Note: Deduct ***A1*** if p is included.

[15 marks]

Examiners report

a. This was on the whole a well answered question and it was rare for a candidate not to obtain full marks on part (a). In part (b) the vast majority of candidates were able to show that the set satisfied the properties of a group apart from associativity which they were also familiar with. Virtually all candidates knew the difference between commutativity and associativity and were able to distinguish between the two. Candidates were familiar with Lagrange's Theorem and many were able to see how it did not apply in the case of this problem. Many candidates found a solution method to part (iii) of the problem and obtained full marks.

b. This was on the whole a well answered question and it was rare for a candidate not to obtain full marks on part (a). In part (b) the vast majority of candidates were able to show that the set satisfied the properties of a group apart from associativity which they were also familiar with. Virtually all candidates knew the difference between commutativity and associativity and were able to distinguish between the two. Candidates were familiar with Lagrange's Theorem and many were able to see how it did not apply in the case of this problem. Many candidates found a solution method to part (iii) of the problem and obtained full marks.

The binary operation $*$ is defined by

$$a * b = a + b - 3 \text{ for } a,\ b \in \mathbb{Z}.$$

The binary operation $\circ$ is defined by

$$a \circ b = a + b + 3 \text{ for } a,\ b \in \mathbb{Z}.$$

Consider the group $\{\mathbb{Z},\ \circ\}$ and the bijection $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(a) = a - 6$.

a. Show that $\{\mathbb{Z},\ *\}$ is an Abelian group. [9]

b. Show that there is no element of order 2. [2]

c. Find a proper subgroup of $\{\mathbb{Z},\ *\}$. [2]

d. Show that the groups $\{\mathbb{Z},\ *\}$ and $\{\mathbb{Z},\ \circ\}$ are isomorphic. [3]

Markscheme

a. closure: $\{\mathbb{Z},\ *\}$ is closed because $a+b-3\in\mathbb{Z}$ ***R1***

identity: $a*e=a+e-3=a$ ***(M1)***

$e=3$ ***A1***

inverse: $a*a^{-1}=a+a^{-1}-3=3$ ***(M1)***

$a^{-1}=6-a$ ***A1***

associative: $a*(b*c)=a*(b+c-3)=a+b+c-6$ ***A1***

$(a\ *\ b)\ *\ c\ =(a\ +\ b\ -\ 3)*\ c\ =\ a\ +\ b\ +\ c\ -\ 6$ ***A1***

associative because $a*(b*c)=(a*b)*c$ ***R1***

$b*a=b+a-3=a+b-3=a*b$ therefore commutative hence Abelian ***R1***

hence $\{\mathbb{Z},\ *\}$ is an Abelian group ***AG***

[9 marks]

b. if a is of order 2 then $a*a=2a-3=3$ therefore $a=3$ ***A1***

which is a contradiction

since $e=3$ and has order 1 ***R1***

Note: ***R1*** for recognising that the identity has order 1.

[2 marks]

c. for example $S=\{-6,\ -3,\ 0,\ 3,\ 6\ldots\}$ or $S=\{\ldots,\ -1,\ 1,\ 3,\ 5,\ 7\ldots\}$ ***A1R1***

Note: ***R1*** for deducing, justifying or verifying that $\{S,*\}$ is indeed a proper subgroup.

[2 marks]

d. we need to show that $f(a*b)=f(a)\circ f(b)$ ***R1***

$f(a*b)=f(a+b-3)=a+b-9$ ***A1***

$f(a)\circ f(b)=(a-6)\circ(b-6)=a+b-9$ ***A1***

hence isomorphic ***AG***

Note: ***R1*** for recognising that f preserves the operation; award ***R1A0A0*** for an attempt to show that $f(a\circ b)=f(a)*f(b)$.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The set S is defined as the set of real numbers greater than 1.

The binary operation $*$ is defined on S by $x * y = (x-1)(y-1)+1$ for all $x,\ y \in S$.

Let $a \in S$.

a. Show that $x * y \in S$ for all $x,\ y \in S$. [2]

b.i. Show that the operation $*$ on the set S is commutative. [2]

b.ii. Show that the operation $*$ on the set S is associative. [5]

c. Show that 2 is the identity element. [2]

d. Show that each element $a \in S$ has an inverse. [3]

Markscheme

a. $x,\ y > 1 \Rightarrow (x-1)(y-1) > 0$ ***M1***

$(x-1)(y-1)+1 > 1$ ***A1***

so $x * y \in S$ for all $x,\ y \in S$ ***AG***

[2 marks]

b.i. $x * y = (x-1)(y-1)+1 = (y-1)(x-1)+1 = y * x$ ***M1A1***

so $*$ is commutative ***AG***

[2 marks]

b.ii. $x * (y * z) = x * ((y-1)(z-1)+1)$ ***M1***

$= (x-1)\left((y-1)(z-1)+1-1\right)+1$ ***(A1)***

$= (x-1)(y-1)(z-1)+1$ ***A1***

$(x * y) * z = ((x-1)(y-1)+1) * z$ ***M1***

$= \left((x-1)(y-1)+1-1\right)(z-1)+1$

$= (x-1)(y-1)(z-1)+1$ ***A1***

so $*$ is associative ***AG***

[5 marks]

c. $2 * x = (2-1)(x-1)+1 = x,\ x * 2 = (x-1)(2-1)+1 = x$ ***M1***

$2 * x = x * 2 = 2\ (2 \in S)$ ***R1***

Note: Accept reference to commutativity instead of explicit expressions.

so 2 is the identity element ***AG***

[2 marks]

d. $a * a^{-1} = 2 \Rightarrow (a-1)(a^{-1}-1)+1 = 2$ ***M1***

so $a^{-1} = 1 + \frac{1}{a-1}$ ***A1***

since $a - 1 > 0 \Rightarrow a^{-1} > 1$ $(a^{-1} * a = a * a^{-1})$ ***R1***

Note: ***R1*** dependent on ***M1***.

so each element, $a \in S$, has an inverse ***AG***

[3 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]
d. [N/A]

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

a. Given that $R = (P \cap Q')'$, list the elements of R . [3]

b. For a set S , let S^* denote the set of all subsets of S , [5]

(i) find P^* ;
(ii) find $n(R^*)$.

Markscheme

a. $P = \{1,\ 2,\ 3\}$
$Q' = \{1,\ 3,\ 5,\ 7,\ 9\}$
so $P \cap Q' = \{1,\ 3\}$ ***(M1)(A1)***
so $(P \cap Q')' = \{2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$ ***A1***
[3 marks]

b. (i) $P^* = \{\{1\},\ \{2\},\ \{3\},\ \{1,\ 2\},\ \{2,\ 3\},\ \{3,\ 1\},\ \{1,\ 2,\ 3),\ \emptyset\}$ ***A2***

Note: Award ***A1*** if one error, ***A0*** for two or more.

(ii) R^* contains: the empty set (1 element); sets containing one element (8 elements); sets containing two elements ***(M1)***
$= \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \end{pmatrix} + \ldots \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ ***(A1)***
$= 2^8$ $(= 256)$ ***A1***
Note: ***FT*** in (ii) applies if no empty set included in (i) and (ii).

[5 marks]

Examiners report

a. This was also a well answered question with many candidates obtaining full marks on both parts of the problem. Some candidates attempted to use a factorial rather than a sum of combinations to solve part (b) (ii) and this led to incorrect answers.

b. This was also a well answered question with many candidates obtaining full marks on both parts of the problem. Some candidates attempted to use a factorial rather than a sum of combinations to solve part (b) (ii) and this led to incorrect answers.

The relation R is defined such that aRb if and only if $4^a - 4^b$ is divisible by 7, where $a,\ b \in \mathbb{Z}^+$.

The equivalence relation S is defined such that cSd if and only if $4^c - 4^d$ is divisible by 6, where $c,\ d \in \mathbb{Z}^+$.

a.i. Show that R is an equivalence relation. [6]

a.ii. Determine the equivalence classes of R. [3]

b. Determine the number of equivalence classes of S. [2]

Markscheme

a.i. **METHOD 1**

reflexive: $4^a - 4^a = 0$ which is divisible by 7 (for all $a \in \mathbb{Z}$) ***R1***

so aRa therefore reflexive

symmetric: Let aRb so that $4^a - 4^b$ is divisible by 7 ***M1***

it follows that $4^b - 4^a = -(4^a - 4^b)$ is also divisible by 7 ***A1***

it follows that bRa therefore symmetric

transitive: let aRb and bRc so that $4^a - 4^b$ and $4^b - 4^c$ are divisible by 7 ***M1***

it follows that $4^a - 4^b = 7M$ and $4^b - 4^c = 7N$ so that $(4^a - 4^b) + (4^b - 4^c) = 4^a - 4^c = 7(M + N)$ ***A1***

therefore aRb and $bRc \Rightarrow aRc$ ***R1***

so that R is transitive

Note: For transitivity, award ***A0*** if the same variable is used to express the multiples of 7; ***R1*** is dependent on the ***M*** mark.

since R R is reflexive, symmetric and transitive, it is an equivalence relation ***AG***

METHOD 2

reflexive: $4^a - 4^a \equiv 0 \bmod 7$ (for all $a \in \mathbb{Z}$) ***R1***

so aRa therefore reflexive

symmetric: let aRb. Then $4^a - 4^b \equiv 0 \bmod 7$ ***M1***

it follows that $4^b - 4^a \equiv -(4^a - 4^b) \equiv 0 \bmod 7$ ***A1***

it follows that bRa therefore symmetric

transitive: let aRb and bRc, *ie*, $4^a - 4^b \equiv 0 \bmod 7$ and $4^b - 4^c \equiv 0 \bmod 7$ ***M1*** so that $4^a - 4^c \equiv (4^a - 4^b) + (4^b - 4^c) \equiv 0 \bmod 7$ ***A1***

therefore aRb and $bRc \Rightarrow aRc$ ***R1***

so R is transitive

Note: For transitivity, award ***A0*** if mod 7 is omitted; ***R1*** is dependent on the ***M*** mark.

since R is reflexive, symmetric and transitive, it is an equivalence relation ***AG***

[6 marks]

a.ii. attempt to solve $4^a \equiv 4 \bmod 7$ or $4^a \equiv 4^2 \equiv 2 \bmod 7$ or $4^a \equiv 4^3 \equiv 1 \bmod 7$ ***(M1)***

the equivalence classes are

$\{1,\ 4,\ 7,\ \ldots\}$, $\{\ 2,\ 5,\ 8,\ \ldots\}$ and $\{3,\ 6,\ 9,\ \ldots\}$ ***A2***

Note: Award ***(M1)A1*** for one or two correct equivalence classes.

[3 marks]

b. starting with 1, we find that 2, 3, 4, … all belong to the same equivalence class **or** $4^c - 4 \equiv 4(4^{c-1} - 1) \equiv 4(2^{c-1} - 1)(2^{c-1} - 1) \equiv 0 \bmod 6$ or $4^c \equiv 4 \bmod 6$ ***(M1)***

therefore there is one equivalence class ***A1***

[2 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

An Abelian group, $\{G,\ *\}$, has 12 different elements which are of the form $a^i * b^j$ where $i \in \{1,\ 2,\ 3,\ 4\}$ and $j \in \{1,\ 2,\ 3\}$. The elements a and b satisfy $a^4 = e$ and $b^3 = e$ where e is the identity.

Let $\{H,\ *\}$ be the proper subgroup of $\{G,\ *\}$ having the maximum possible order.

a. State the possible orders of an element of $\{G,\ *\}$ and for each order give an example of an element of that order. [8]

b. (i) State a generator for $\{H,\ *\}$. [7]

(ii) Write down the elements of $\{H,\ *\}$.

(iii) Write down the elements of the coset of H containing a.

Markscheme

a. orders are 1 2 3 4 6 12 ***A2***

Note: ***A1*** for four or five correct orders.

Note: For the rest of this question condone absence of xxx and accept equivalent expressions.

order:	1	element:	2	$A1$
	2		a^2	$A1$
	3		b or b^2	$A1$
	4		a or a^3	$A1$
	6		$a^2 * b$ or $a^2 * b^2$	$A1$
	12		$a * b$ or $a * b^2$ or $a^3 * b$ or $a^3 * b^2$	$A1$

[8 marks]

b. (i) H has order 6 ***(R1)***

generator is $a^2 * b$ or $a^2 * b^2$ ***A1***

(ii) $H = \{e,\ a^2 * b,\ b^2,\ a^2,\ b,\ a^2 * b^2\}$ ***A3***

Note: ***A2*** for 4 or 5 correct. ***A1*** for 2 or 3 correct.

(iii) required coset is Ha (or aH) ***(R1)***

$Ha = \{a,\ a^3 * b,\ a * b^2,\ a^3,\ a * b,\ a^3 * b^2\}$ ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]

The relation R is defined such that xRy if and only if $|x| + |y| = |x + y|$ for $x,\ y,\ y \in \mathbb{R}$.

a.i. Show that R is reflexive. [2]

a.ii. Show that R is symmetric. [2]

b. Show, by means of an example, that R is not transitive. [4]

Markscheme

a.i. (for $x \in \mathbb{R}$), $|x| + |x| = 2\,|x|$ ***A1***

and $|x| + |x| = |2x| = 2\,|x|$ ***A1***

hence xRx

so R is reflexive ***AG***

Note: Award ***A1*** for correct verification of identity for $x > 0$; ***A1*** for correct verification for $x \le 0$.

[2 marks]

a.ii.if $xRy \Rightarrow |x| + |y| = |x + y|$

$|x| + |y| = |y| + |x|$ ***A1***

$|x + y| = |y + x|$ ***A1***

hence yRx

so R is symmetric ***AG***

[2 marks]

b. recognising a condition where transitivity does not hold ***(M1)***

(*eg*, $x > 0$, $y = 0$ and $z < 0$)

for example, $1R0$ and $0R(-1)$ ***A1***

however $|1| + |-1| \neq |1 + -1|$ ***A1***

so $1R(-1)$ (for example) is not true ***R1***

hence R is not transitive ***AG***

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

The group G has a unique element, h , of order 2.

(i) Show that ghg^{-1} has order 2 for all $g \in G$.

(ii) Deduce that $gh = hg$ for all $g \in G$.

Markscheme

(i) consider $(ghg^{-1})^2$ ***M1***

$= ghg^{-1}ghg^{-1} = gh^2g^{-1} = gg^{-1} = e$ ***A1***

ghg^{-1} cannot be order 1 (= e) since h is order 2 ***R1***

so ghg^{-1} has order 2 ***AG***

(ii) but h is the unique element of order 2 ***R1***

hence $ghg^{-1} = h \Rightarrow gh = hg$ ***A1AG***

[5 marks]

Examiners report

This question was by far the problem to be found most challenging by the candidates. Many were able to show that ghg^{-1} had order one or two although hardly any candidates also showed that the order was not one thus losing a mark. Part a (ii) was answered correctly by a few candidates who noticed the equality of h and ghg^{-1}. However, many candidates went into algebraic manipulations that led them nowhere and did not justify any marks. Part (b) (i) was well answered by a small number of students who appreciated the nature of the identity and element h thus forcing the other two elements to have order four. However, (ii) was only occasionally answered correctly and even in these cases not systematically. It is possible that candidates lacked time to fully explore the problem. A small number of candidates "guessed" the correct answer.

Two functions, F and G , are defined on $A = \mathbb{R}\backslash\{0,\ 1\}$ by

$$F(x) = \frac{1}{x},\ G(x) = 1 - x,\ \text{for all } x \in A.$$

(a) Show that under the operation of composition of functions each function is its own inverse.

(b) F and G together with four other functions form a closed set under the operation of composition of functions.

Find these four functions.

Markscheme

(a) the following two calculations show the required result

$F \circ F(x) = \frac{1}{\frac{1}{x}} = x$

$G \circ G(x) = 1 - (1 - x) = x$ ***M1A1A1***

[3 marks]

(b) part (a) shows that the identity function defined by $I(x) = x$ belongs to S ***A1***

the two compositions of F and G are:

$F \circ G(x) = \frac{1}{1-x}$; ***(M1)A1***

$G \circ F(x) = 1 - \frac{1}{x}\left(= \frac{x-1}{x}\right)$ ***(M1)A1***

the final element is

$G \circ F \circ G(x) = 1 - \frac{1}{1-x}\left(= \frac{x}{x-1}\right)$ ***(M1)A1***

[7 marks]

Total [10 marks]

Examiners report

This question was generally well done. In part(a), the quickest answer involved showing that squaring the function gave the identity. Some candidates went through the more elaborate method of finding the inverse function in each case.

The binary operation $*$ is defined for $x,\ y \in S = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ by

$$x * y = (x^3y - xy) \bmod 7.$$

a. Find the element e such that $e * y = y$, for all $y \in S$. [2]

b. (i) Find the least solution of $x * x = e$. [5]

(ii) Deduce that $(S,\ *)$ is not a group.

c. Determine whether or not e is an identity element. [3]

Markscheme

a. attempt to solve $e^3y - ey \equiv y \bmod 7$ ***(M1)***

the only solution is $e = 5$ ***A1***

[2 marks]

b. (i) attempt to solve $x^4 - x^2 \equiv 5 \bmod 7$ ***(M1)***

least solution is $x = 2$ ***A1***

(ii) suppose $(S,\ *)$ is a group with order 7 ***A1***

2 has order 2 ***A1***

since 2 does not divide 7, Lagrange's Theorem is contradicted ***R1***

hence, $(S,\ *)$ is not a group ***AG***

[5 marks]

c. (5 is a left-identity), so need to test if it is a right-identity:

ie, is $y * 5 = y$? ***M1***

$1 * 5 = 0 \ne 1$ ***A1***

so 5 is not an identity ***A1***

[3 marks]

Total [10 marks]

Examiners report

a. Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

b. Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested

for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

c. Many candidates were not sufficiently familiar with modular arithmetic to complete this question satisfactorily. In particular, some candidates completely ignored the requirement that solutions were required to be found modulo 7, and returned decimal answers to parts (a) and (b). Very few candidates invoked Lagrange's theorem in part (b)(ii). Some candidates were under the misapprehension that a group had to be Abelian, so tested for commutativity in part (b)(ii). It was pleasing that many candidates realised that an identity had to be both a left and right identity.

All of the relations in this question are defined on $\mathbb{Z}\backslash\{0\}$.

a. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow x+y>7$ is [4]

(i) reflexive;
(ii) symmetric;
(iii) transitive.

b. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow -2<x-y<2$ is [4]

(i) reflexive;
(ii) symmetric;
(iii) transitive.

c. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow xy>0$ is [4]

(i) reflexive;

(ii) symmetric;

(iii) transitive.

d. Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow \frac{x}{y} \in \mathbb{Z}$ is [4]

(i) reflexive;
(ii) symmetric;
(iii) transitive.

e. One of the relations from parts (a), (b), (c) and (d) is an equivalence relation. [3]

For this relation, state what the equivalence classes are.

Markscheme

a. (i) not reflexive *e.g.* $1 + 1 = 2$ ***R1***

(ii) symmetric since $x + y = y + x$ ***R1***

(iii) *e.g.* $1 + 11 > 7$, $11 + 2 > 7$ but $1 + 2 = 3$, so not transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

b. (i) reflexive since $x-x=0$ ***R1***

(ii) symmetric since $|x-y| = |y-x|$ ***R1***

(iii) *e.g.* 1R2, 2R3 but $1 - 3 = -2$, so not transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

c. (i) reflexive since $x^2 > 0$ ***R1***

(ii) symmetric since $xy = yx$ ***R1***

(iii) $xy > 0$ and $yz > 0 \Rightarrow xy^2z > 0 \Rightarrow xz > 0$ since $y^2 > 0$, so transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

d. (i) reflexive since $\frac{x}{x} = 1$ ***R1***

(ii) not symmetric *e.g.* $\frac{2}{1} = 2$ but $\frac{1}{2} = 0.5$ ***R1***

(iii) $\frac{x}{y} \in \mathbb{Z}$ and $\frac{y}{z} \in \mathbb{Z} \Rightarrow \frac{xy}{yz} = \frac{x}{z} \in \mathbb{Z}$, so transitive ***M1A1***

Note: For each ***R1*** mark the correct decision and a valid reason must be given.

[4 marks]

e. only (c) is an equivalence relation ***(A1)***

the equivalence classes are

$\{1, 2, 3,\ldots\}$ and $\{-1,-2,-3,\ldots\}$ ***A1A1***

[3 marks]

Examiners report

a. Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

b. Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

c. Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

d. Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

e. Generally this question was well answered, with students showing a sound knowledge of relations. There were a few candidates who mixed reflexive and symmetric qualities and marks were also lost because reasoning was either unclear or absent. Most students were able to offer counterexamples for transitivity in parts (a) and (b) but a number lost marks in failing to give adequate working to show transitivity in parts (c) and (d). That said, there were a pleasing number of good solutions here showing all the required rigour. Whilst most students were able to identify part (c) as an equivalence relation, surprisingly few gave the correct equivalence classes.

Let $A = \{a,\ b\}$.

Let the set of all these subsets be denoted by $P(A)$. The binary operation symmetric difference, Δ , is defined on $P(A)$ by $X \Delta Y = (X \backslash Y) \cup (Y \backslash X)$ where X , $Y \in P(A)$.

Let $\mathbb{Z}_4 = \{0,\ 1,\ 2,\ 3\}$ and $+_4$ denote addition modulo 4.

Let S be any non-empty set. Let $P(S)$ be the set of all subsets of S . For the following parts, you are allowed to assume that Δ, $\cup$ and $\cap$ are associative.

a. Write down all four subsets of A . [1]

b. Construct the Cayley table for $P(A)$ under Δ . [3]

c. Prove that $\{P(A),\ \Delta\}$ is a group. You are allowed to assume that Δ is associative. [3]

d. Is $\{P(A), \Delta\}$ isomorphic to $\{\mathbb{Z}_4,\ +_4\}$? Justify your answer. [2]

e. (i) State the identity element for $\{P(S), \Delta\}$. [4]

(ii) Write down X^{-1} for $X \in P(S)$.

(iii) Hence prove that $\{P(S), \Delta\}$ is a group.

f. Explain why $\{P(S), \cup\}$ is not a group. [1]

g. Explain why $\{P(S), \cap\}$ is not a group. [1]

Markscheme

a. $\emptyset$, { a} , { b} , { a, b} ***A1***

[1 mark]

b.

Δ	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\varnothing$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a, b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	$\varnothing$	$\{a\}$
$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a\}$	$\varnothing$

A3

Note: Award ***A2*** for one error, ***A1*** for two errors, ***A0*** for more than two errors.

[3 marks]

c. closure is seen from the table above ***A1***

$\emptyset$ is the identity ***A1***

each element is self-inverse ***A1***

Note: Showing each element has an inverse is sufficient.

associativity is assumed so we have a group ***AG***

[3 marks]

d. not isomorphic as in the above group all elements are self-inverse whereas in $(\mathbb{Z}_4,\ +_4)$ there is an element of order 4 (*e.g.* 1) ***R2***

[2 marks]

e.

(i) $\emptyset$ is the identity ***A1***

(ii) $X^{-1} = X$ ***A1***

(iii) if X and Y are subsets of S then $X \Delta Y$ (the set of elements that belong to X or Y but not both) is also a subset of S, hence closure is proved ***R1***

$\{P(S), \Delta\}$ is a group because it is closed, has an identity, all elements have inverses (and Δ is associative) ***R1AG***

[4 marks]

f. not a group because although the identity is $\emptyset$, if $X \neq \emptyset$ it is impossible to find a set Y such that $X \cup Y = \emptyset$, so there are elements without an inverse ***R1AG***

[1 mark]

g. not a group because although the identity is S, if $X \neq S$ is impossible to find a set Y such that $X \cap Y = S$, so there are elements without an inverse ***R1AG***

[1 mark]

Examiners report

a. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

b. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

c. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

d. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

e. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

f. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

g. A surprising number of candidates were unable to answer part (a) and consequently were unable to access much of the rest of the question. Most candidates however, were successful in parts (a), (b) and (c), and it was pleasing to see the preparedness of candidates in these parts. There were also many good answers for parts (d) and (e) although the third part of (e) caused the most problems with candidates failing to provide sufficient reasoning. Few candidates managed good responses to parts (f) and (g).

The binary operation $*$ is defined on the set $T = \{0,\ 2,\ 3,\ 4,\ 5,\ 6\}$ by $a * b = (a + b - ab)(\bmod 7),\ a,\ b \in T$.

a. Copy and complete the following Cayley table for $\{T,\ *\}$. [4]

*	0	2	3	4	5	6
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3	6				
4	4	5				
5	5	4				
6	6	3				

b. Prove that $\{T,\ *\}$ forms an Abelian group. [7]

c. Find the order of each element in T. [4]

d. Given that $\{H,\ *\}$ is the subgroup of $\{T,\ *\}$ of order 2, partition T into the left cosets with respect to H. [3]

Markscheme

a. Cayley table is

*	0	2	3	4	5	6
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3	6	4	2	0	5
4	4	5	2	6	3	0
5	5	4	0	3	6	2
6	6	3	5	0	2	4

A4

award ***A4*** for all 16 correct, ***A3*** for up to 2 errors, ***A2*** for up to 4 errors, ***A1*** for up to 6 errors

[4 marks]

b. closed as no other element appears in the Cayley table ***A1***

symmetrical about the leading diagonal so commutative ***R1*** hence it is Abelian

0 is the identity

as $x * 0 (= 0 * x) = x + 0 - 0 = x$ ***A1***

0 and 2 are self inverse, 3 and 5 is an inverse pair, 4 and 6 is an inverse pair ***A1***

Note: Accept "Every row and every column has a 0 so each element has an inverse".

$(a * b) * c = (a + b - ab) * c = a + b - ab + c - (a + b - ab)c$ ***M1***

$= a + b + c - ab - ac - bc + abc$ ***A1***

$a * (b * c) = a * (b + c - bc) = a + b + c - bc - a(b + c - bc)$ ***A1***

$= a + b + c - ab - ac - bc + abc$

so $(a * b) * c = a * (b * c)$ and $*$ is associative

Note: Inclusion of mod 7 may be included at any stage.

[7 marks]

c. 0 has order 1 and 2 has order 2 ***A1***

$3^2 = 4,\ 3^3 = 2,\ 3^4 = 6,\ 3^5 = 5,\ 3^6 = 0$ so 3 has order 6 ***A1***

$4^2 = 6,\ 4^3 = 0$ so 4 has order 3 ***A1***

5 has order 6 and 6 has order 3 ***A1***

[4 marks]

d. $H = \{0,\ 2\}$ ***A1***

$0 * \{0,\ 2\} = \{0,\ 2\},\ 2 * \{0,\ 2\} = \{2,\ 0\},\ 3 * \{0,\ 2\} = \{3,\ 6\},\ 4 * \{0,\ 2\} = \{4,\ 5\},$

$5 * \{0,\ 2\} = \{5,\ 4\},\ 6 * \{0,\ 2\} = \{6,\ 3\}$ ***M1***

Note: Award the ***M1*** if sufficient examples are used to find at least two of the cosets.

so the left cosets are $\{0,\ 2\},\ \{3,\ 6\},\ \{4,\ 5\}$ ***A1***

[3 marks]

Total [18 marks]

Examiners report

a.

b. [N/A]

c. [N/A]

d. [N/A]

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $f(x,\ y) = (2x^3 + y^3,\ x^3 + 2y^3)$.

a. Show that f is a bijection. [12]

b. Hence write down the inverse function $f^{-1}(x,\ y)$. [1]

Markscheme

a. for f to be a bijection it must be both an injection and a surjection ***R1***

Note: Award this ***R1*** for stating this anywhere.

suppose that $f(a,\ b) = f(c,\ d)$ ***(M1)***

it follows that

$2a^3 + b^3 = 2c^3 + d^3$ and $a^3 + 2b^3 = c^3 + 2d^3$ ***A1***

attempting to solve the two equations ***M1***

to obtain $3a^3 = 3c^3$

Note: Award ***M1*** only if a good attempt is made to solve the system.

$\Rightarrow a = c$ and therefore $b = d$ ***A1***

f is an injection because $f(a,\ b) = f(c,\ d) \Rightarrow (a,\ b) = (c,\ d)$ ***R1***

Note: Award this ***R1*** for stating this anywhere providing that an attempt is made to prove injectivity.

let $(p,\ q) \in \mathbb{R} \times \mathbb{R}$ and let $f(r,\ s) = (p,\ q)$ ***(M1)***

then, $p = 2r^3 + s^3$ and $q = r^3 + 2s^3$ ***A1***

attempting to solve the two equations ***M1***

$r = \sqrt[3]{\frac{2p-q}{3}}$ and $s = \sqrt[3]{\frac{2q-p}{3}}$ ***A1A1***

f is a surjection because given $(p,\ q) \in \mathbb{R} \times \mathbb{R}$, there exists $(r,\ s) \in \mathbb{R} \times \mathbb{R}$ such that $f(r,\ s) = (p,\ q)$ ***R1***

Note: Award this ***R1*** for stating this anywhere providing that an attempt is made to prove surjectivity.

[12 marks]

b. $\left(f^{-1}(x,\ y) =\right)\ \left(\sqrt[3]{\frac{2x-y}{3}}, \sqrt[3]{\frac{2y-x}{3}}\right)$ ***A1***

Note: ***A1*** for correct expressions in x and y, allow ***FT*** only if the expression is deduced in part (a).

[1 mark]

Examiners report

a. [N/A]
b. [N/A]

Let A be the set $\{x|x \in \mathbb{R},\ x \neq 0\}$. Let B be the set $\{x|x \in]-1,\ +1[,\ x \neq 0\}$.

A function $f : A \to B$ is defined by $f(x) = \frac{2}{\pi}\arctan(x)$.

Let D be the set $\{x|x \in \mathbb{R},\ x > 0\}$.

A function $g : \mathbb{R} \to D$ is defined by $g(x) = \mathrm{e}^x$.

a. (i) Sketch the graph of $y = f(x)$ and hence justify whether or not f is a bijection. [13]

(ii) Show that A is a group under the binary operation of multiplication.

(iii) Give a reason why B is not a group under the binary operation of multiplication.

(iv) Find an example to show that $f(a \times b) = f(a) \times f(b)$ is not satisfied for all $a,\ b \in A$.

b. (i) Sketch the graph of $y = g(x)$ and hence justify whether or not g is a bijection. [8]

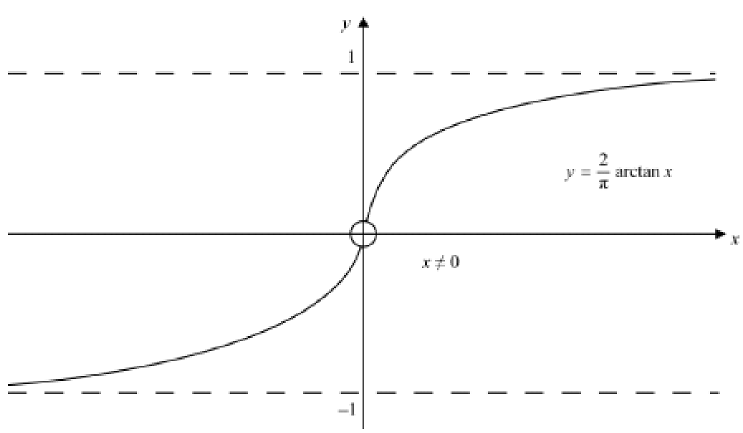
(ii) Show that $g(a + b) = g(a) \times g(b)$ for all $a,\ b \in \mathbb{R}$.

(iii) Given that $\{\mathbb{R},\ +\}$ and $\{D,\ \times\}$ are both groups, explain whether or not they are isomorphic.

Markscheme

a.

(i)



A1

Notes: Award ***A1*** for general shape, labelled asymptotes, and showing that $x \neq 0$.

graph shows that it is injective since it is increasing or by the horizontal line test ***R1***

graph shows that it is surjective by the horizontal line test ***R1***

Note: Allow any convincing reasoning.

so f is a bijection ***A1***

(ii) closed since non-zero real times non-zero real equals non-zero real ***A1R1*** we know multiplication is associative ***R1***

identity is 1 ***A1***

inverse of x is $\frac{1}{x} (x \neq 0)$ ***A1***

hence it is a group ***AG***

(iii) B does not have an identity ***A2***

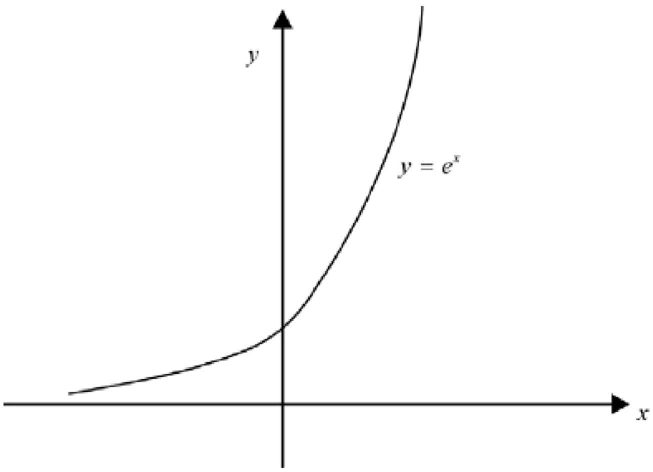
hence it is not a group ***AG***

(iv) $f(1 \times 1) = f(1) = \frac{1}{2}$ whereas $f(1) \times f(1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ is one counterexample ***A2***

hence statement is not satisfied ***AG***

[13 marks]

b.



award ***A1*** for general shape going through (0, 1) and with domain $\mathbb{R}$ ***A1***

graph shows that it is injective since it is increasing or by the horizontal line test and graph shows that it is surjective by the horizontal line test ***R1***

Note: Allow any convincing reasoning.

so g is a bijection ***A1***

(ii) $g(a+b) = \mathrm{e}^{a+b}$ and $g(a) \times g(b) = \mathrm{e}^a \times \mathrm{e}^b = \mathrm{e}^{a+b}$ ***M1A1***

hence $g(a+b) = g(a) \times g(b)$ ***AG***

(iii) since g is a bijection and the homomorphism rule is obeyed ***R1R1***

the two groups are isomorphic ***A1***

[8 marks]

Examiners report

a. [N/A]

b. [N/A]

(a) Show that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x,\ y) = (2x + y,\ x - y)$ is a bijection.

(b) Find the inverse of f.

Markscheme

(a) we need to show that the function is both injective and surjective to be a bijection ***R1***

suppose $f(x,\ y) = f(u,\ v)$ ***M1***

$(2x + y,\ x - y) = (2u + v,\ u - v)$

forming a pair of simultaneous equations ***M1***

$2x + y = 2u + v$ (i)

$x - y = u - v$ (ii)

$(i) + (ii) \Rightarrow 3x = 3u \Rightarrow x = u$ ***A1***

$(i) - 2(ii) \Rightarrow 3y = 3v \Rightarrow y = v$ ***A1***

hence function is injective ***R1***

let $2x + y = s$ and $x - y = t$ ***M1***

$\Rightarrow 3x = s + t$

$\Rightarrow x = \frac{s+t}{3}$ ***A1***

also $3y = s - 2t$

$\Rightarrow y = \frac{s-2t}{3}$ ***A1***

for any $(s,\ t) \in \mathbb{R} \times \mathbb{R}$ there exists $(x,\ y) \in \mathbb{R} \times \mathbb{R}$ and the function is surjective ***R1***

[10 marks]

(b) the inverse is $f^{-1}(x,\ y) = \left(\frac{x+y}{3},\ \frac{x-2y}{3}\right)$ ***A1***

[1 mark]

Total [11 marks]

Examiners report

Many students were able to show that the expression was injective, but found more difficulty in showing it was subjective. As with question 1 part (e), a number of candidates did not realise that the answer to part (b) came directly from part (a), hence the reason for it being worth only one mark.

The binary operation $*$ is defined on $\mathbb{R}$ as follows. For any elements a , $b \in \mathbb{R}$

$$a * b = a + b + 1.$$

a. (i) Show that $*$ is commutative. [5]

(ii) Find the identity element.

(iii) Find the inverse of the element a .

b. The binary operation $\cdot$ is defined on $\mathbb{R}$ as follows. For any elements a , $b \in \mathbb{R}$ [6]

$a \cdot b = 3ab$. The set S is the set of all ordered pairs $(x,\ y)$ of real numbers and the binary operation $\odot$ is defined on the set S as $(x_1,\ y_1) \odot (x_2,\ y_2) = (x_1 * x_2,\ y_1 \cdot y_2)$.

Determine whether or not $\odot$ is associative.

Markscheme

a. (i) if $*$ is commutative $a * b = b * a$

since $a + b + 1 = b + a + 1$, $*$ is commutative ***R1***

(ii) let e be the identity element

$a * e = a + e + 1 = a$ ***M1***

$\Rightarrow e = -1$ ***A1***

(iii) let a have an inverse, a^{-1}

$a * a^{-1} = a + a^{-1} + 1 = -1$ ***M1***

$\Rightarrow a^{-1} = -2 - a$ ***A1***

[5 marks]

b. $(x_1,\ y_1) \odot ((x_2,\ y_2) \odot (x_3,\ y_3)) = (x_1,\ y_1) \odot (x_2 + x_3 + 1,\ 3y_2y_3)$ ***M1***

$= (x_1 + x_2 + x_3 + 2,\ 9y_1y_2y_3)$ ***A1A1***

$((x_1,\ y_1) \odot (x_2,\ y_2)) \odot (x_3,\ y_3) = (x_1 + x_2 + 1,\ 3y_1y_2) \odot (x_3,\ y_3)$ ***M1***

$= (x_1 + x_2 + x_3 + 2,\ 9y_1y_2y_3)$ ***A1***

hence $\odot$ is associative ***R1***

[6 marks]

Examiners report

a. Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates.

b. Part (b) was completed by many candidates, but a significant number either did not understand what was meant by associative, confused associative with commutative, or were unable to complete the algebra.

(a) Draw the Cayley table for the set of integers $G = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6, $+_6$.

(b) Show that $\{G,\ +_6\}$ is a group.

(c) Find the order of each element.

(d) Show that $\{G,\ +_6\}$ is cyclic and state its generators.

(e) Find a subgroup with three elements.

(f) Find the other proper subgroups of $\{G,\ +_6\}$.

Markscheme

(a)

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

A3

Note: Award ***A2*** for 1 error, ***A1*** for 2 errors and ***A0*** for more than 2 errors.

[3 marks]

(b) The table is closed ***A1***

Identity element is 0 ***A1***

Each element has a unique inverse (0 appears exactly once in each row and column) ***A1***

Addition mod 6 is associative ***A1***

Hence $\{G, +_6\}$ forms a group ***AG***

[4 marks]

(c) 0 has order 1 (0 = 0),

1 has order 6 (1 + 1 + 1 + 1 + 1 + 1 = 0),

2 has order 3 (2 + 2 + 2 = 0),

3 has order 2 (3 + 3 = 0),

4 has order 3 (4 + 4 + 4 = 0),

5 has order 6 (5 + 5 + 5 + 5 + 5 + 5 = 0). ***A3***

Note: Award ***A2*** for 1 error, ***A1*** for 2 errors and ***A0*** for more than 2 errors.

[3 marks]

(d) Since 1 and 5 are of order 6 (the same as the order of the group) every element can be written as sums of either 1 or 5. Hence the group is cyclic. ***R1***

The generators are 1 and 5. ***A1***

[2 marks]

(e) A subgroup of order 3 is $(\{0, 2, 4\}, +_6)$ ***A2***

Note: Award ***A1*** if only {0, 2, 4} is seen.

[2 marks]

(f) Other proper subgroups are $(\{0\}+_6)$, $(\{0, 3\}+_6)$ ***A1A1***

Note: Award ***A1*** if only $\{0\}$, $\{0, 3\}$ is seen.

[2 marks]

Total [16 marks]

Examiners report

The table was well done as was showing its group properties. The order of the elements in (b) was done well except for the order of 0 which was often not given. Finding the generators did not seem difficult but correctly stating the subgroups was not often done. The notion of a 'proper' subgroup is not well known.

The function $f : [0,\ \infty[\to [0,\ \infty[$ is defined by $f(x) = 2\mathrm{e}^{x} + \mathrm{e}^{-x} - 3$.

(a) Find $f'(x)$.

(b) Show that *f* is a bijection.

(c) Find an expression for $f^{-1}(x)$.

Markscheme

(a) $f'(x) = 2\mathrm{e}^{x} - \mathrm{e}^{-x}$ ***A1***

[1 mark]

(b) *f* is an injection because $f'(x) > 0$ for $x \in [0,\ \infty[$ ***R2***

(accept GDC solution backed up by a correct graph)

since $f(0) = 0$ and $f(x) \to \infty$ as $x \to \infty$, (and *f* is continuous) it is a surjection ***R1***

hence it is a bijection ***AG***

[3 marks]

(c) let $y = 2\mathrm{e}^{x} + \mathrm{e}^{-x} - 3$ ***M1***

so $2\mathrm{e}^{2x} - (y + 3)\mathrm{e}^{x} + 1 = 0$ ***A1***

$\mathrm{e}^{x} = \frac{y+3\pm\sqrt{(y+3)^{2}-8}}{4}$ ***A1***

$x = \ln\left(\frac{y+3\pm\sqrt{(y+3)^{2}-8}}{4}\right)$ ***A1***

since $x \geqslant 0$ we must take the positive square root ***(R1)***

$f^{-1}(x) = \ln\left(\frac{x+3+\sqrt{(x+3)^{2}-8}}{4}\right)$ ***A1***

[6 marks]

Total [10 marks]

Examiners report

In many cases the attempts at showing that *f* is a bijection were unconvincing. The candidates were guided towards showing that *f* is an injection by noting that $f'(x) > 0$ for all *x*, but some candidates attempted to show that $f(x) = f(y) \Rightarrow x = y$ which is much more difficult. Solutions to (c) were often disappointing, with the algebra defeating many candidates.

The universal set contains all the positive integers less than 30. The set *A* contains all prime numbers less than 30 and the set *B* contains all positive integers of the form $3 + 5n$ $(n \in \mathbb{N})$ that are less than 30. Determine the elements of

a. $A \setminus B$; [4]

b. $A \Delta B$. [3]

Markscheme

a. $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ ***(A1)***

$B = \{3, 8, 13, 18, 23, 28\}$ ***(A1)***

Note: *FT* on their *A* and *B*

$A \setminus B = \{$elements in *A* that are not in *B*$\}$ ***(M1)***

$= \{2, 5, 7, 11, 17, 19, 29\}$ ***A1***

[4 marks]

b. $B \setminus A = \{8, 18, 28\}$ ***(A1)***

$A \Delta B = (A \setminus B) \cup (B \setminus A)$ ***(M1)***

$= \{2, 5, 7, 8, 11, 17, 18, 19, 28, 29\}$ ***A1***

[3 marks]

Examiners report

a. It was disappointing to find that many candidates wrote the elements of A and B incorrectly. The most common errors were the inclusion of 1 as a prime number and the exclusion of 3 in B. It has been suggested that some candidates use *N* to denote the positive integers. If this is the case, then it is important to emphasise that the IB notation is that *N* denotes the positive integers and zero and IB candidates should all be aware of that. Most candidates solved the remaining parts of the question correctly and follow through ensured that those candidates with incorrect A and/or B were not penalised any further.

b. It was disappointing to find that many candidates wrote the elements of A and B incorrectly. The most common errors were the inclusion of 1 as a prime number and the exclusion of 3 in B. It has been suggested that some candidates use *N* to denote the positive integers. If this is the case, then it is important to emphasise that the IB notation is that *N* denotes the positive integers and zero and IB candidates should all be aware of that. Most candidates solved the remaining parts of the question correctly and follow through ensured that those candidates with incorrect A and/or B were not penalised any further.

The binary operation $*$ is defined on $\mathbb{N}$ by $a * b = 1 + ab$.

Determine whether or not $*$

a. is closed; [2]

b. is commutative; [2]

c. is associative; [3]

d. has an identity element. [3]

Markscheme

a. $*$ is closed ***A1***

because $1 + ab \in \mathbb{N}$ (when $a, b \in \mathbb{N}$) ***R1***

[2 marks]

b. consider

$a * b = 1 + ab = 1 + ba = b * a$ ***M1A1***

therefore $*$ is commutative

[2 marks]

c. **EITHER**

$a * (b * c) = a * (1 + bc) = 1 + a(1 + bc)$ $(= 1 + a + abc)$ ***A1***

$(a * b) * c = (1 + ab) * c = 1 + c(1 + ab)$ $(= 1 + c + abc)$ ***A1***

(these two expressions are unequal when $a \neq c$) so $*$ is not associative ***R1***

OR

proof by counter example, for example

$1 * (2 * 3) = 1 * 7 = 8$ ***A1***

$(1 * 2) * 3 = 3 * 3 = 10$ ***A1***

(these two numbers are unequal) so $*$ is not associative ***R1***

[3 marks]

d. let e denote the identity element; so that

$a * e = 1 + ae = a$ gives $e = \frac{a-1}{a}$ (where $a \neq 0$) ***M1*** then any valid statement such as: $\frac{a-1}{a} \notin \mathbb{N}$ or e is not unique ***R1***

there is therefore no identity element ***A1***

Note: Award the final ***A1*** only if the previous ***R1*** is awarded.

[3 marks]

Examiners report

a. For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

b. For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

c. For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

d. For the commutative property some candidates began by setting $a * b = b * a$. For the identity element some candidates confused $e * a$ and ea stating $ea = a$. Others found an expression for an inverse element but then neglected to state that it did not belong to the set of natural numbers or that it was not unique.

A group with the binary operation of multiplication modulo 15 is shown in the following Cayley table.

$\times_{15}$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	a	b	c
13	13	11	7	1	14	d	e	f
14	14	13	11	8	7	g	h	i

a. Find the values represented by each of the letters in the table. [3]

b. Find the order of each of the elements of the group. [3]

c. Write down the three sets that form subgroups of order 2. [2]

d. Find the three sets that form subgroups of order 4. [4]

Markscheme

a. $a = 1 \quad b = 8 \quad c = 4$

$d = 8 \quad e = 4 \quad f = 2$

$g = 4 \quad h = 2 \quad i = 1$ ***A3***

Note: Award ***A3*** for 9 correct answers, ***A2*** for 6 or more, and ***A1*** for 3 or more.

[3 marks]

b.

Elements	Order
1	1
4, 11, 14	2
2, 7, 8, 13	4

A3

Note: Award ***A3*** for 8 correct answers, ***A2*** for 6 or more, and ***A1*** for 4 or more.

[3 marks]

c. $\{1,\ 4\},\ \{1,\ 11\},\ \{1,\ 14\}$ ***A1A1***

Note: Award ***A1*** for 1 correct answer and ***A2*** for all 3 (and no extras).

[2 marks]

d. $\{1,\ 2,\ 4,\ 8\},\ \{1,\ 4,\ 7,\ 13\},$ ***A1A1***

$\{1,\ 4,\ 11,\ 14\}$ ***A2***

[4 marks]

Total [12 marks]

Examiners report

a. The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

b. The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

c. The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

d. The first two parts of this question were generally well done. It was surprising to see how many difficulties there were with parts (c) and (d) with many answers given as {4}, {11} and {14} for example.

---

a. Given that p , q and r are elements of a group, prove the left-cancellation rule, *i.e.* $pq = pr \Rightarrow q = r$. [4]

Your solution should indicate which group axiom is used at each stage of the proof.

b. Consider the group G , of order 4, which has distinct elements a , b and c and the identity element e . [10]

(i) Giving a reason in each case, explain why ab cannot equal a or b .

(ii) Given that c is self inverse, determine the two possible Cayley tables for G .

(iii) Determine which one of the groups defined by your two Cayley tables is isomorphic to the group defined by the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ under multiplication of complex numbers. Your solution should include a correspondence between a, b, c, e and $1, -1, \mathrm{i}, -\mathrm{i}$.

Markscheme

a. $pq = pr$

$p^{-1}(pq) = p^{-1}(pr)$, every element has an inverse ***A1***

$(p^{-1}p)q = (p^{-1}p)r$, Associativity ***A1***

Note: Brackets in lines 2 and 3 must be seen.

$eq = er, p^{-1}p = e$, the identity ***A1***

$q = r$, $ea = a$ for all elements a of the group ***A1***

[4 marks]

b. (i) let $ab = a$ so $b = e$ be which is a contradiction ***R1***

let $ab = b$ so $a = e$ which is a contradiction ***R1***

therefore ab cannot equal either a or b ***AG***

(ii) the two possible Cayley tables are

table 1

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

A2

table 2

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

A2

(iii) the group defined by table 1 is isomorphic to the given group ***R1***

because

EITHER

both contain one self-inverse element (other than the identity) ***R1***

OR

both contain an inverse pair ***R1***

OR

both are cyclic ***R1***

THEN

the correspondence is $e \to 1, c \to -1, a \to i, b \to -i$

(or vice versa for the last two) ***A2***

Note: Award the final ***A2*** only if the correct group table has been identified.

[10 marks]

Examiners report

a. Solutions to (a) were often poor with inadequate explanations often seen. It was not uncommon to see $pq = pr$

$p^{-1}pq = p^{-1}pr$

$q = r$

without any mention of associativity. Many candidates understood what was required in (b)(i), but solutions to (b)(ii) were often poor with the tables containing elements such as *ab* and *bc* without simplification. In (b)(iii), candidates were expected to determine the isomorphism by noting that the group defined by {1, –1, i, –i} under multiplication is cyclic or that –1 is the only self-inverse element apart from the identity, without necessarily writing down the Cayley table in full which many candidates did. Many candidates just stated that there was a bijection between the two groups without giving any justification for this.

b. Solutions to (a) were often poor with inadequate explanations often seen. It was not uncommon to see $pq = pr$

$p^{-1}pq = p^{-1}pr$

$q = r$

without any mention of associativity. Many candidates understood what was required in (b)(i), but solutions to (b)(ii) were often poor with the tables containing elements such as *ab* and *bc* without simplification. In (b)(iii), candidates were expected to determine the isomorphism by noting that the group defined by {1, –1, i, –i} under multiplication is cyclic or that –1 is the only self-inverse element apart from the identity, without necessarily writing down the Cayley table in full which many candidates did. Many candidates just stated that there was a bijection between the two groups without giving any justification for this.

A binary operation is defined on $\{-1, 0, 1\}$ by

$$A \odot B = \begin{cases} -1, & \text{if } |A| < |B| \\ 0, & \text{if } |A| = |B| \\ 1, & \text{if } |A| > |B| \,. \end{cases}$$

(a) Construct the Cayley table for this operation.

(b) Giving reasons, determine whether the operation is

(i) closed;

(ii) commutative;

(iii) associative.

Markscheme

(a) the Cayley table is

$$\begin{array}{c c} & \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$ ***M1A2***

Notes: Award ***M1*** for setting up a Cayley table with labels.

Deduct ***A1*** for each error or omission.

[3 marks]

(b) (i) closed ***A1***

because all entries in table belong to $\{-1, 0, 1\}$ ***R1***

(ii) not commutative ***A1***

because the Cayley table is not symmetric, or counter-example given ***R1***

(iii) not associative ***A1***

for example because ***M1***

$0 \odot (-1 \odot 0) = 0 \odot 1 = -1$

but

$(0 \odot -1) \odot 0 = -1 \odot 0 = 1$ ***A1***

or alternative counter-example

[7 marks]

Total [10 marks]

Examiners report

This question was generally well done, with the exception of part(b)(iii), showing that the operation is non-associative.

Sets X and Y are defined by $X =]0,\ 1[;\ Y = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$.

a. (i) Sketch the set $X \times Y$ in the Cartesian plane. [5]

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$.

b. Consider the function $f : X \times Y \to \mathbb{R}$ defined by $f(x,\ y) = x + y$ and the function $g : X \times Y \to \mathbb{R}$ defined by $g(x,\ y) = xy$. [10]

(i) Find the range of the function f.

(ii) Find the range of the function g.

(iii) Show that f is an injection.

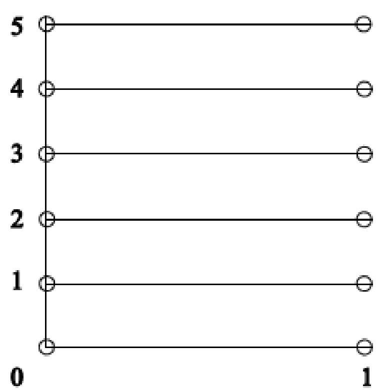
(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x,\ y) = \frac{1}{2}$.

Markscheme

a.

(i)

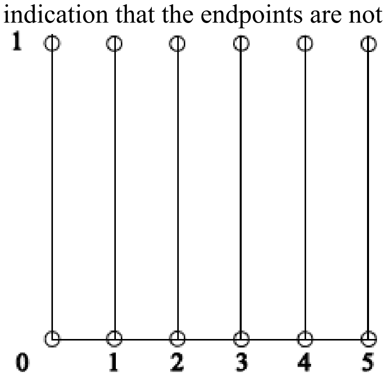


correct horizontal lines *A1*

correctly labelled axes *A1*

clear indication that the endpoints are not included *A1*

(ii)



fully correct diagram *A1*

Note: Do not penalize the inclusion of endpoints twice.

(iii) the intersection is empty *A1*

[5 marks]

b. (i) range $(f) =]0,\ 1[\cup]1,\ 2[\cup \mathrm{L} \cup]5,\ 6[$ *A1A1*

Note: *A1* for six intervals and *A1* for fully correct notation.

Accept $0 < x < 6,\ x \neq 0, 1, 2, 3, 4, 5, 6$.

(ii) range $(g) = [0,\ 5[$ ***A1***

(iii) Attempt at solving

$f(x_1,\ y_1) = f(x_2,\ y_2)$ ***M1***

$f(x,\ y) \in]y,\ y+1[\Rightarrow y_1 = y_2$ ***M1***

and then $x_1 = x_2$ ***A1***

so f is injective ***AG***

(iv) $f^{-1}(\pi) = (\pi - 3,\ 3)$ ***A1A1***

(v) solutions: (0.5, 1), (0.25, 2), $\left(\frac{1}{6},\ 3\right)$, (0.125, 4), (0.1, 5) ***A2***

Note: ***A2*** for all correct, ***A1*** for 2 correct.

[10 marks]

Examiners report

a. [N/A]

b. [N/A]

Let $f : G \to H$ be a homomorphism of finite groups.

a. Prove that $f(e_G) = e_H$, where e_G is the identity element in G and e_H is the identity element in H. [2]

b. (i) Prove that the kernel of $f,\ K = \mathrm{Ker}(f)$, is closed under the group operation. [6]

(ii) Deduce that K is a subgroup of G.

c. (i) Prove that $gkg^{-1} \in K$ for all $g \in G,\ k \in K$. [6]

(ii) Deduce that each left coset of K in G is also a right coset.

Markscheme

a. $f(g) = f(e_G g) = f(e_G)f(g)$ for $g \in G$ ***M1A1***

$\Rightarrow f(e_G) = e_H$ ***AG***

[2 marks]

b. (i) closure: let k_1 and $k_2 \in K$, then $f(k_1k_2) = f(k_1)f(k_2)$ ***M1A1***

$= e_He_H = e_H$ ***A1***

hence $k_1k_2 \in K$ ***R1***

(ii) K is non-empty because e_G belongs to K ***R1***

a closed non-empty subset of a finite group is a subgroup ***R1AG***

[6 marks]

c. (i) $f(gkg^{-1}) = f(g)f(k)f(g^{-1})$ ***M1***

$= f(g)e_Hf(g^{-1}) = f(gg^{-1})$ ***A1***

$= f(e_G) = e_H$ ***A1***

$\Rightarrow gkg^{-1} \in K$ ***AG***

(ii) clear definition of both left and right cosets, seen somewhere. ***A1***

use of part (i) to show $gK \subseteq Kg$ ***M1***

similarly $Kg \subseteq gK$ ***A1***

hence $gK = Kg$ ***AG***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Let X and Y be sets. The functions $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f$ is the identity function on X.

a. Prove that: [6]

(i) f is an injection,

(ii) g is a surjection.

b. Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions f and g to show that g is not necessarily a bijection. [3]

Markscheme

a. (i) to test injectivity, suppose $f(x_1) = f(x_2)$ ***M1***

apply g to both sides $g\left(f(x_1)\right) = g\left(f(x_2)\right)$ ***M1***

$\Rightarrow x_1 = x_2$ ***A1***

so f is injective ***AG***

Note: Do not accept arguments based on "f has an inverse".

(ii) to test surjectivity, suppose $x \in X$ ***M1***

define $y = f(x)$ ***M1***

then $g(y) = g\left(f(x)\right) = x$ A1

so g is surjective ***AG***

[6 marks]

b. choose, for example, $f(x) = \sqrt{x}$ and $g(y) = y^2$ ***A1***

then $g \circ f(x) = \left(\sqrt{x}\right)^2 = x$ ***A1***

the function g is not injective as $g(x) = g(-x)$ ***R1***

[3 marks]

Total [9 marks]

Examiners report

a. Those candidates who formulated the questions in terms of the basic definitions of injectivity and surjectivity were usually sucessful. Otherwise, verbal attempts such as 'f is one - to - one $\Rightarrow f$ is injective' or 'g is surjective because its range equals its codomain', received no credit. Some candidates made the false assumption that f and g were mutual inverses.

b. Few candidates gave completely satisfactory answers. Some gave functions satisfying the mutual identity but either not defined on the given sets or for which g was actually a bijection.

Let $(H, \ *)$ be a subgroup of the group $(G, \ *)$.

Consider the relation R defined in G by xRy if and only if $y^{-1} * x \in H$.

(a) Show that R is an equivalence relation on G.

(b) Determine the equivalence class containing the identity element.

Markscheme

(a) R is reflexive as $x^{-1} * x = e \in H \Rightarrow xRx$ for any $x \in G$ ***A1***

if xRy then $y^{-1} * x = h \in H$

but $h \in H \Rightarrow h^{-1} \in H$, *ie*, $\underbrace{(y^{-1} * x)^{-1}}_{x^{-1} * y} \in H$ ***M1***

therefore yRx

R is symmetric ***A1***

if xRy then $y^{-1} * x = h \in H$ and if yRz then $z^{-1} * y = k \in H$ ***M1***

$k * h \in H$, *ie*, $\underbrace{(z^{-1} * y) * (y^{-1} * x)}_{z^{-1} * x} \in H$ ***A1***

therefore xRz

R is transitive ***A1***

so R is an equivalence relation on G ***AG***

[6 marks]

(b) $xRe \Leftrightarrow e^{-1} * x \in H$ ***M1***

$x \in H$ ***A1***

$[e] = H$ ***A1*** ***N0***

[3 marks]

Examiners report

Part (a) was fairly well answered by many candidates. They knew how to apply the equivalence relations axioms in this particular example. Part (b) however proved to be very challenging and hardly any correct answers were seen.

Consider the set A consisting of all the permutations of the integers $1, 2, 3, 4, 5$.

a. Two members of A are given by $p = (1\ 2\ 5)$ and $q = (1\ 3)(2\ 5)$. [4]

Find the single permutation which is equivalent to $q \circ p$.

b. State a permutation belonging to A of order [3]

(i) 4;

(ii) 6.

c. Let $P =$ {all permutations in A where exactly two integers change position}, [4]

and $Q =$ {all permutations in A where the integer 1 changes position}.

(i) List all the elements in $P \cap Q$.

(ii) Find $n(P \cap Q')$.

Markscheme

a. $q \circ p = (1\ 3)(2\ 5)(1\ 2\ 5)$ ***(M1)***

$= (1\ 5\ 3)$ ***M1A1A1***

Note: ***M1*** for an answer consisting of disjoint cycles, ***A1*** for $(1\ 5\ 3)$,

A1 for either (2) or (2) omitted.

Note: Allow $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

If done in the wrong order and obtained $(1\ 3\ 2)$, award ***A2.***

[4 marks]

b. (i) any cycle with length 4 *eg* (1234) ***A1***

(ii) any permutation with 2 disjoint cycles one of length 2 and one of length 3 *eg* $(1\ 2)\ (3\ 4\ 5)$ ***M1A1***

Note: Award ***M1A0*** for any permutation with 2 non-disjoint cycles one of length 2 and one of length 3.

Accept non cycle notation.

[3 marks]

c. (i) $(1,\ 2)$, $(1,\ 3)$, $(1,\ 4)$, $(1,\ 5)$ ***M1A1***

(ii) $(2\ 3)$, $(2\ 4)$, $(2\ 5)$, $(3\ 4)$, $(3\ 5)$, $(4\ 5)$ ***(M1)***

6 ***A1***

Note: Award ***M1*** for at least one correct cycle.

[4 marks]

Total [11 marks]

Examiners report

a. Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

b. Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

c. Many students were unable to start the question, seemingly as they did not understand the cyclic notation. Many of those that did understand found it quite straightforward to obtain good marks on this question.

Given the sets A and B, use the properties of sets to prove that $A \cup (B' \cup A)' = A \cup B$, justifying each step of the proof.

Markscheme

$A \cup (B' \cup A)' = A \cup (B \cap A')$ De Morgan ***M1A1***

$= (A \cup B) \cap (A \cup A')$ Distributive property ***M1A1***

$= (A \cup B) \cap U$ (Union of set and its complement) ***A1***

$= A \cup B$ (Intersection with the universal set) ***AG***

Note: Do not accept proofs using Venn diagrams unless the properties are clearly stated.

Note: Accept double inclusion proofs: ***M1A1*** for each inclusion, final ***A1*** for conclusion of equality of sets.

[5 marks]

Examiners report

[N/A]

(a) Write down why the table below is a Latin square.

$$\begin{array}{c|c} & \begin{matrix} d & e & b & a & c \end{matrix} \\ \begin{matrix} d \\ e \\ b \\ a \\ c \end{matrix} & \begin{bmatrix} c & d & e & b & a \\ d & e & b & a & c \\ a & b & d & c & e \\ b & a & c & e & d \\ e & c & a & d & b \end{bmatrix} \end{array}$$

(b) Use Lagrange's theorem to show that the table is not a group table.

Markscheme

(a) Each row and column contains all the elements of the set. ***A1A1***

[2 marks]

(b) There are 5 elements therefore any subgroup must be of an order that is a factor of 5 ***R2***

But there is a subgroup $\begin{array}{cc} & \begin{array}{cc} e & a \end{array} \\ \begin{array}{c} e \\ a \end{array} & \begin{pmatrix} e & a \\ a & e \end{pmatrix} \end{array}$ of order 2 so the table is not a group table ***R2***

Note: Award ***R0R2*** for "a is an element of order 2 which does not divide the order of the group".

[4 marks]

Total [6 marks]

Examiners report

Part (a) presented no problem but finding the order two subgroups (Lagrange's theorem was often quoted correctly) was beyond some candidates.

Possibly presenting the set in non-alphabetical order was the problem.

Let $p = 2^k + 1,\ k \in \mathbb{Z}^+$ be a prime number and let G be the group of integers 1, 2, ..., $p - 1$ under multiplication defined modulo p.

By first considering the elements $2^1,\ 2^2,\ ...,\ 2^k$ and then the elements $2^{k+1},\ 2^{k+2},\ \ldots$, show that the order of the element 2 is $2k$.

Deduce that $k = 2^n$ for $n \in \mathbb{N}$.

Markscheme

The identity is 1. ***(R1)***

Consider

$2^1,\ 2^2,\ 2^3,\ ...,\ 2^k$

$2^k = p - 1$ ***R1***

Therefore all the above powers of two are different ***R1***

Now consider

$2^{k+1} \equiv 2p - 2(\bmod p) = p - 2$ ***M1A1***

$2^{k+2} \equiv 2p - 4(\bmod p) = p - 4$ ***A1***

$2^{k+3} = p - 8$

etc.

$2^{2k-1} = p - 2^{k-1}$

$2^{2k} = p - 2^k$ ***A1***

$= 1$ ***A1***

and this is the first power of 2 equal to 1. ***R2***

The order of 2 is therefore $2k$. ***AG***

Using Lagrange's Theorem, it follows that $2k$ is a factor of 2^k , the order of the group, in which case k must be as given. ***R2***

[12 marks]

Examiners report

Few solutions were seen to this question with many candidates unable even to start.

Prove that $(A \cap B)\backslash(A \cap C) = A \cap (B\backslash C)$ where A, B and C are three subsets of the universal set U.

Markscheme

$(A \cap B)\backslash(A \cap C) = (A \cap B) \cap (A \cap C)'$ ***M1***

$= (A \cap B) \cap (A' \cup C')$ ***A1***

$= (A \cap B \cap A') \cup (A \cap B \cap C')$ ***A1***

$= (A \cap A' \cap B) \cup (A \cap B \cap C')$ ***A1***

$= (\emptyset \cap B) \cup (A \cap B \cap C')$ ***(A1)***

$= \emptyset \cup (A \cap B \cap C')$

$= (A \cap (B \cap C'))$ ***A1***

$= A \cap (B\backslash C)$ ***AG***

Note: Do not accept proofs by Venn diagram.

[6 marks]

Examiners report

Venn diagram 'proof' are not acceptable. Those who used de Morgan's laws usually were successful in this question.

Let $\{G,\ *\}$ be a finite group and let H be a non-empty subset of G . Prove that $\{H,\ *\}$ is a group if H is closed under $*$.

Markscheme

the associativity property carries over from G ***R1***

closure is given ***R1***

let $h \in H$ and let n denote the order of h, (this is finite because G is finite) ***M1***

it follows that $h^n = e$, the identity element ***R1***

and since H is closed, $e \in H$ ***R1***

since $h * h^{n-1} = e$ ***M1***

it follows that h^{n-1} is the inverse, h^{-1}, of h ***R1***

and since H is closed, $h^{-1} \in H$ so each element of H has an inverse element ***R1*** the four requirements for H to be a group are therefore satisfied ***AG***

[8 marks]

Examiners report

[N/A]

The group $\{G,\ *\}$ has identity e_G and the group $\{H,\ \circ\}$ has identity e_H. A homomorphism f is such that $f : G \to H$. It is given that $f(e_G) = e_H$.

a. Prove that for all $a \in G,\ f(a^{-1}) = (f(a))^{-1}$. [4]

b. Let $\{H,\ \circ\}$ be the cyclic group of order seven, and let p be a generator. [4]

Let $x \in G$ such that $f(x) = p^2$.

Find $f(x^{-1})$.

c. Given that $f(x * y) = p$, find $f(y)$. [4]

Markscheme

a. $f(e_G) = e_H \Rightarrow f(a * a^{-1}) = e_H$ ***M1***

f is a homomorphism so $f(a * a^{-1}) = f(a) \circ f(a^{-1}) = e_H$ ***M1A1***

by definition $f(a) \circ (f(a))^{-1} = e_H$ so $f(a^{-1}) = (f(a))^{-1}$ (by the left-cancellation law) ***R1***

[4 marks]

b. from (a) $f(x^{-1}) = (f(x))^{-1}$

hence $f(x^{-1}) = (p^2)^{-1} = p^5$ ***M1A1***

[2 marks]

c. $f(x * y) = f(x) \circ f(y)$ (homomorphism) ***(M1)***

$p^2 \circ f(y) = p$ ***A1***

$f(y) = p^5 \circ p$ ***(M1)***

$= p^6$ ***A1***

[4 marks]

Total [10 marks]

Examiners report

a. Part (a) was well answered by those who understood what a homomorphism is. However many candidates simply did not have this knowledge and consequently could not get into the question.

b. Part (b) was well answered, even by those who could not do (a). However, there were many who having not understood what a homomorphism is, made no attempt on this easy question part. Understandably many lost a mark through not simplifying p^{-2} to p^5.

c. Those who knew what a homomorphism is generally obtained good marks in part (c).

H and K are subgroups of a group G. By considering the four group axioms, prove that $H \cap K$ is also a subgroup of G.

Markscheme

closure: let $a,\ b \in H \cap K$, so that $a,\ b \in H$ and $a,\ b \in K$ ***M1***

therefore $ab \in H$ and $ab \in K$ so that $ab \in H \cap K$ ***A1***

associativity: this carries over from G ***R1***

identity: the identity $e \in H$ and $e \in K$ ***M1***

therefore $e \in H \cap K$ ***A1***

inverse:

$a \in H \cap K$ implies $a \in H$ and $a \in K$ ***M1***

it follows that $a^{-1} \in H$ and $a^{-1} \in K$ ***A1***

and therefore that $a^{-1} \in H \cap K$ ***A1***

the four group axioms are therefore satisfied ***AG***

[8 marks]

Examiners report

This question presented the most difficulty for students. Overall the candidates showed a lack of ability to present a formal proof. Some gained points for the proof of the identity element in the intersection and the statement that the associative property carries over from the group. However, the vast majority gained no points for the proof of closure or the inverse axioms.

Prove that set difference is not associative.

Markscheme

we are trying to prove $(A\backslash B)\backslash C \neq A\backslash(B\backslash C)$ ***M1(A1)***

$\text{LHS} = (A \cap B')\backslash C$ ***(A1)***

$= (A \cap B') \cap C'$ ***A1***

$\text{RHS} = A\backslash(B \cap C')$

$= A \cap (B \cap C')'$ ***(A1)***

$= A \cap (B' \cup C)$ ***A1***

as LHS does not contain any element of C and RHS does, LHS $\neq$ RHS ***R1***

hence set difference is not associative ***AG***

Note: Accept answers which use a proof containing a counter example.

Total [7 marks]

Examiners report

This question was found difficult by a large number of candidates, but a number of correct solutions were seen. A number of candidates who understood what was required failed to gain the final reasoning mark. Many candidates seemed to be ill-prepared to deal with this style of question.

Define $f : \mathbb{R} \backslash \{0.5\} \to \mathbb{R}$ by $f(x) = \frac{4x+1}{2x-1}$.

a. Prove that f is an injection. [4]

b. Prove that f is not a surjection. [4]

Markscheme

a. **METHOD 1**

$f(x) = f(y) \Rightarrow \frac{4x+1}{2x-1} = \frac{4y+1}{2y-1}$ ***M1A1***

for attempting to cross multiply and simplify ***M1***

$(4x+1)(2y-1) = (2x-1)(4y+1)$

$\Rightarrow 8xy + 2y - 4x - 1 = 8xy + 2x - 4y - 1 \Rightarrow 6y = 6x$

$\Rightarrow x = y$ ***A1***

hence an injection ***AG***

METHOD 2

$f'(x) = \frac{4(2x-1)-2(4x+1)}{(2x-1)^2} = \frac{-6}{(2x-1)^2}$ ***M1A1***

$< 0 \quad (\text{for all } x \neq 0.5)$ ***R1***

therefore the function is decreasing on either side of the discontinuity

and $f(x) < 2$ and $x < 0.5$ for $f(x) > 0.5$ ***R1***

hence an injection ***AG***

Note: If a correct graph of the function is shown, and the candidate states this is decreasing in each part (or horizontal line test) and hence an injection, award ***M1A1R1***.

[4 marks]

b. **METHOD 1**

attempt to solve $y = \frac{4x+1}{2x-1}$ ***M1***

$y(2x-1) = 4x+1 \Rightarrow 2xy - y = 4x+1$ ***A1***

$2xy - 4x = 1 + y \Rightarrow x = \frac{1+y}{2y-4}$ ***A1***

no value for $y = 2$ ***R1***

hence not a surjection ***AG***

METHOD 2

consider $y = 2$ ***A1***

attempt to solve $2 = \frac{4x+1}{2x-1}$ ***M1***

$4x - 2 = 4x + 1$ ***A1***

which has no solution ***R1***

hence not a surjection ***AG***

Note: If a correct graph of the function is shown, and the candidate states that because there is a horizontal asymptote at $y = 2$ then the function is not a surjection, award ***M1R1***.

[4 marks]

Total [8 marks]

Examiners report

a. Most students indicated an understanding of the concepts of Injection and Surjection, but many did not give rigorous proofs. Even where graphs were used, it was very common for a sketch to be so imprecise with no asymptotes marked that it was difficult to award even partial credit. Some candidates mistakenly stated that the function was not surjective because 0.5 was not in the domain.

b. Most students indicated an understanding of the concepts of Injection and Surjection, but many did not give rigorous proofs. Even where graphs were used, it was very common for a sketch to be so imprecise with no asymptotes marked that it was difficult to award even partial credit. Some candidates mistakenly stated that the function was not surjective because 0.5 was not in the domain.

Consider the sets

$$G = \left\{ \frac{n}{6^i} | n \in \mathbb{Z},\ i \in \mathbb{N} \right\},\ H = \left\{ \frac{m}{3^j} | m \in \mathbb{Z},\ j \in \mathbb{N} \right\}.$$

a. Show that $(G,\ +)$ forms a group where $+$ denotes addition on $\mathbb{Q}$. Associativity may be assumed. [5]

b. Assuming that $(H,\ +)$ forms a group, show that it is a proper subgroup of $(G,\ +)$. [4]

c. The mapping $\phi : G \to G$ is given by $\phi(g) = g + g$, for $g \in G$. [7]

Prove that ϕ is an isomorphism.

Markscheme

a. closure: $\frac{n_1}{6^{i_1}} + \frac{n_2}{6^{i_2}} = \frac{6^{i_2}n_1 + 6^{i_1}n_2}{6^{i_1+i_2}} \in G$ ***A1R1***

Note: Award ***A1*** for RHS of equation. ***R1*** is for the use of two different, but not necessarily most general elements, and the result $\in G$ or equivalent.

identity: 0 ***A1***

inverse: $\frac{-n}{6^i}$ ***A1***

since associativity is given, $(G, +)$ forms a group ***R1AG***

Note: The ***R1*** is for considering closure, the identity, inverses and associativity.

[5 marks]

b. it is required to show that H is a proper subset of G ***(M1)***

let $\frac{n}{3^i} \in H$ ***M1***

then $\frac{n}{3^i} = \frac{2^i n}{6^i} \in G$ hence H is a subgroup of G ***A1***

$H \neq G$ since $\frac{1}{6} \in G$ but $\frac{1}{6} \notin H$ ***A1***

Note: The final ***A1*** is only dependent on the first ***M1***.

hence, H is a proper subgroup of G ***AG***

[4 marks]

c. consider $\phi(g_1 + g_2) = (g_1 + g_2) + (g_1 + g_2)$ ***M1***

$= (g_1 + g_1) + (g_2 + g_2) = \phi(g_1) + \phi(g_2)$ ***A1***

(hence ϕ is a homomorphism)

injectivity: let $\phi(g_1) = \phi(g_2)$ ***M1***

working within $\mathbb{Q}$ we have $2g_1 = 2g_2 \Rightarrow g_1 = g_2$ ***A1***

surjectivity: considering even and odd numerators ***M1***

$\phi\left(\frac{n}{6^i}\right) = \frac{2n}{6^i}$ and $\phi\left(\frac{3(2n+1)}{6^{i+1}}\right) = \frac{2n+1}{6^i}$ ***A1A1***

hence ϕ is an isomorphism ***AG***

[7 marks]

Total [16 marks]

Examiners report

a. This part was generally well done. Where marks were lost, it was usually because a candidate failed to choose two different elements in the proof of closure.

b. Only a few candidates realised that they did not have to prove that H is a group - that was stated in the question. Some candidates tried to invoke Lagrange's theorem, even though G is an infinite group.

c. Many candidates showed that the mapping is injective. Most attempts at proving surjectivity were unconvincing. Those candidates who attempted to establish the homomorphism property sometimes failed to use two different elements.

Consider the following functions

$f :]1,\ +\infty[\to \mathbb{R}^+$ where $f(x) = (x-1)(x+2)$
$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $g(x,\ y) = (\sin(x+y),\ x+y)$
$h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $h(x,\ y) = (x+3y,\ 2x+y)$

(a) Show that f is bijective.

(b) Determine, with reasons, whether

(i) g is injective;

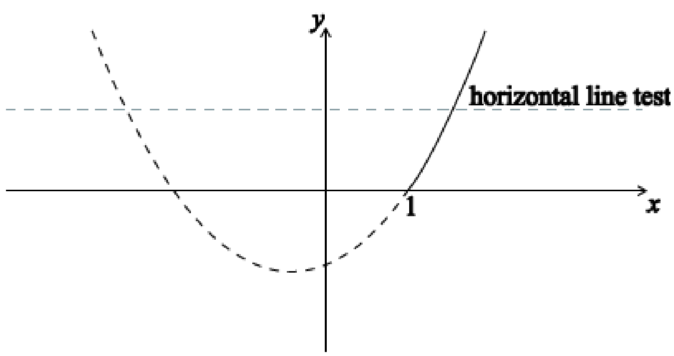
(ii) g is surjective.

(c) Find an expression for $h^{-1}(x,\ y)$ and hence justify that h has an inverse function.

Markscheme

(a) **Method 1**

sketch of the graph of f ***(M1)***



range of f = co-domain, therefore f is surjective ***R1***
graph of f passes the horizontal line test, therefore f is injective ***R1***
therefore f is bijective ***AG***

Note: Other explanations may be given (*eg* use of derivative or description of parabola).

Method 2
Injective: $f(a) = f(b) \Rightarrow a = b$ ***M1***
$(a-1)(a+2) = (b-1)(b+2)$
$a^2 + a = b^2 + b$
solving for a by completing the square, or the quadratic formula, ***A1***
$a = b$
surjective: for all $y \in \mathbb{R}^+$ there exists $x \in]1,\ \infty[$ such that $f(x) = y$
solving $y = x^2 + x - 2$ for x, $x = \frac{\sqrt{4y+9}-1}{2}$. For all positive real y, the minimum value for $\sqrt{4y+9}$ is 3. Hence, $x \geqslant 1$ ***R1***
since f is both injective and surjective, f is bijective. ***AG***
Method 3
f is bijective if and only if f has an inverse ***(M1)***
solving $y = x^2 + x - 2$ for $x,\ x = \frac{\sqrt{4y+9}-1}{2}$. For all positive real y, the minimum value for $\sqrt{4y+9}$ is 3. Hence, $x \geqslant 1$ ***R1***

$f^{-1}(x) = \frac{\sqrt{4x+9}-1}{2}$ ***R1***

f has an inverse, hence f is bijective ***AG***

[3 marks]

(b) (i) attempt to find counterexample ***(M1)***

eg $g(x,\ y) = g(y,\ x),\ x \neq y$ ***A1***

g is not injective ***R1***

(ii) $-1 \leqslant \sin(x+y) \leqslant 1$ ***(M1)***

range of g is $[-1,\ 1] \times \mathbb{R} \neq \mathbb{R} \times \mathbb{R}$ ***A1***

g is not surjective ***R1***

[6 marks]

(c) let $h(x,\ y) = (u,\ v)$

then $u = x + 3y$

$v = 2x + y$ ***(M1)***

solving simultaneous equations ***(M1)***

eg $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$x = \frac{-u+3v}{5} y = \frac{2u-v}{5}$ ***A1***

hence $h^{-1}(x,\ y) = \left(\frac{-x+3y}{5},\ \frac{2x-y}{5}\right)$ ***A1***

as this expression is defined for any values of $(x,\ y) \in \mathbb{R} \times \mathbb{R}$ ***R1***

the inverse of h exists ***AG***

[5 marks]

Examiners report

For part (a), given the command term ‘show that’ and the number of marks for this part, the best approach is a graphical one, i.e., an informal approach. Many candidates chose an algebraic approach and generally made correct statements for injective and surjective. However, they often did not follow through with the necessary algebraic manipulation to make a valid conclusion. In part (b), many candidates were not able to provide valid counter-examples. In part (c) It was obvious that quite a few candidates had not seen this type of function before. Those that were able to find the inverse generally did not justify their result, and hence could not earn the final R mark.

---

a. Let $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ f(m,\ x) = (-1)^m x$. Determine whether *f* is [4]

(i) surjective;

(ii) injective.

b. P is the set of all polynomials such that $P = \left\{\sum\limits_{i=0}^{n} a_i x^i | n \in \mathbb{N}\right\}$. [4]

Let $g : P \to P,\ g(p) = xp$. Determine whether g is

(i) surjective;

(ii) injective.

c. Let $h : \mathbb{Z} \to \mathbb{Z}^+$, $h(x) = \left\{\begin{array}{cc} 2x, & x > 0 \\ 1 - 2x, & x \leqslant 0 \end{array}\right\}$. Determine whether h is [7]

(i) surjective;

(ii) injective.

Markscheme

a. (i) let $x \in \mathbb{R}$

for example, $f(0,\ x) = x$, ***M1***

hence f is surjective ***A1***

(ii) for example, $f(2,\ 3) = f(4,\ 3) = 3,$ but $(2,\ 3) \neq (4,\ 3)$ ***M1***

hence f is not injective ***A1***

[4 marks]

b. (i) there is no element of P such that $g(p) = 7$, for example ***R1***

hence g is not surjective ***A1***

(ii) $g(p) = g(q) \Rightarrow xp = xq \Rightarrow p = q$, hence g is injective ***M1A1***

[4 marks]

c. (i) for $x > 0,\ h(x) = 2,\ 4,\ 6,\ 8\ldots$ ***A1***

for $x \leqslant 0,\ h(x) = 1,\ 3,\ 5,\ 7\ldots$ ***A1***

therefore h is surjective ***A1***

(ii) for $h(x) = h(y)$, since an odd number cannot equal an even number, there are only two possibilities: ***R1***

$x,\ y > 0,\ 2x = 2y \Rightarrow x = y;$ ***A1***

$x,\ y \leqslant 0,\ 1 - 2x = 1 - 2y \Rightarrow x = y$ ***A1***

therefore h is injective ***A1***

Note: This can be demonstrated in a variety of ways.

[7 marks]

Examiners report

a. This was the least successfully answered question on the paper. Candidates often could quote the definitions of surjective and injective, but often could not apply the definitions in the examples.

a) Some candidates failed to show convincingly that the function was surjective, and not injective.

b. This was the least successfully answered question on the paper. Candidates often could quote the definitions of surjective and injective, but often could not apply the definitions in the examples.

b) Some candidates had trouble interpreting the notation used in the question, hence could not answer the question successfully.

c. This was the least successfully answered question on the paper. Candidates often could quote the definitions of surjective and injective, but often could not apply the definitions in the examples.

c) Many candidates failed to appreciate that the function is discrete, and hence erroneously attempted to differentiate the function to show that it is monotonic increasing, hence injective. Others who provided a graph again showed a continuous rather than discrete function.

The function $f:\mathbb{R}\to\mathbb{R}$ is defined as $f: x \to \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$.

a. Prove that f is [4]

(i) not injective;

(ii) not surjective.

b. The relation R is defined for $a,\ b \in \mathbb{R}$ so that aRb if and only if $f(a)\times f(b)=1$. [8]

Show that R is an equivalence relation.

c. The relation R is defined for $a,\ b \in \mathbb{R}$ so that aRb if and only if $f(a)\times f(b)=1$. [2]

State the equivalence classes of R.

Markscheme

a. (i) *eg* $f(2)=f(3)$ ***M1***

hence $f(a)=f(b)\Rightarrow a=b$ ***R1***

so not injective ***AG***

(ii) *eg* Codomain is $\mathbb{R}$ and range is $\{-1,\ 1\}$ ***M1***

these not the same so not surjective ***R1AG***

Note: if counter example is given it must be stated it is not in the range to obtain the ***R1***. *Eg* $f(x)=2$ has no solution as $f(x)\in\{-1,\ 1\}\forall x$.

[4 marks]

b. if $a \geq 0$ then $f(a)\times f(a)=1\times 1=1$ ***A1***

if $a<0$ then $f(a)\times f(a)=-1\times -1=1$ ***A1***

in either case aRa so R is reflexive ***R1***

$aRb \Rightarrow f(a)\times f(b)=1\Rightarrow f(b)\times f(a)=1\Rightarrow bRa$ ***A1***

so R is symmetric ***R1***

if aRb then either $a\geq 0$ and $b \geq 0$ or $a<0$ and $b<0$

if $a\geq 0$ and $b\geq 0$ and bRc then $c\geq 0$ so $f(a)\times f(c)=1\times 1=1$ and aRc ***A1***

if $a<0$ and $b<0$ and bRc then $c<0$ so $f(a)\times f(c)=-1\times -1=1$ and aRc ***A1***

in either case aRb and $bRc\Rightarrow aRc$ so R is transitive ***R1***

Note: Accept

$f(a)\times f(b)\times f(b)\times f(c)=1\times 1=1\Rightarrow f(a)\times 1\times\ f(c)=1\Rightarrow\ f(a)\times f(c)=1$

Note: for each property just award ***R1*** if at least one of the A marks is awarded.

as R is reflexive, symmetric and transitive it is an equivalence relation ***AG***

[8 marks]

c. equivalence classes are $[0,\ \infty[$ and $]-\infty,\ 0[$ ***A1A1***

Note: Award ***A1A0*** for both intervals open.

[2 marks]

Total [14 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x,\ y) = \left(xy^2,\ \frac{x}{y}\right)$.

Show that *f* is a bijection.

Markscheme

for *f* to be a bijection it must be both an injection and a surjection ***R1***

Note: Award this ***R1*** for stating this anywhere.

injection:

let $f(a, b) = f(c,\ d)$ so that ***(M1)***

$ab^2 = cd^2$ and $\frac{a}{b} = \frac{c}{d}$ ***A1***

dividing the equations,

$b^3 = d^3$ so $b = d$ ***A1***

substituting,

$a = c$ ***A1***

it follows that *f* is an injection because $f(a, b) = f(c,\ d) \Rightarrow (a, b) = (c,\ d)$ ***R1***

surjection:

let $f(a, b) = (c,\ d)$ where $(c,\ d) \in \mathbb{R}^+ \times \mathbb{R}^+$ ***(M1)***

then $c = ab^2$ and $d = \frac{a}{b}$ ***A1***

dividing,

$b^3 = \frac{c}{d}$ so $b = \sqrt[3]{\frac{c}{d}}$ ***A1***

substituting,

$a = d \times \sqrt[3]{\frac{c}{d}}$ ***A1***

it follows that *f* is a surjection because

given $(c,\ d) \in \mathbb{R}^+ \times \mathbb{R}^+$, there exists $(a,\ b) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $f(a, b) = (c,\ d)$ ***R1***

therefore *f* is a bijection ***AG***

[11 marks]

Examiners report

Candidates who knew that they were required to give a rigorous demonstration that *f* was injective and surjective were generally successful, although the formality that is needed in this style of demonstration was often lacking. Some candidates, however, tried unsuccessfully to give a verbal explanation or even a 2-D version of the horizontal line test. In 2-D, the only reliable method for showing that a function *f* is injective is to show that $f(a, b) = f(c,\ d) \Rightarrow (a, b) = (c,\ d)$.

Let G be a finite cyclic group.

(a) Prove that G is Abelian.

(b) Given that a is a generator of G, show that a^{-1} is also a generator.

(c) Show that if the order of G is five, then all elements of G, apart from the identity, are generators of G.

Markscheme

(a) let a be a generator and consider the (general) elements $b = a^m,\ c = a^n$ ***M1***

then

$bc = a^m a^n$ ***A1***

$= a^n a^m$ (using associativity) ***R1***

$= cb$ ***A1***

therefore G is Abelian ***AG***

[4 marks]

(b) let G be of order p and let $m \in \{1, \ldots\ldots, p\}$, let a be a generator

consider $aa^{-1} = e \Rightarrow a^m(a^{-1})^m = e$ ***M1R1***

this shows that $(a^{-1})^m$ is the inverse of a^m ***R1***

as m increases from 1 to p, a^m takes p different values and it generates G ***R1***

it follows from the uniqueness of the inverse that $(a^{-1})^m$ takes p different values and is a generator ***R1***

[5 marks]

(c) **EITHER**

by Lagrange, the order of any element divides the order of the group, *i.e.* 5 ***R1***

the only numbers dividing 5 are 1 and 5 ***R1***

the identity element is the only element of order 1 ***R1***

all the other elements must be of order 5 ***R1***

so they all generate G ***AG***

OR

let a be a generator.

successive powers of a and therefore the elements of G are

$a,\ a^2,\ a^3,\ a^4$ and $a^5 = e$ ***A1***

successive powers of a^2 are $a^2,\ a^4,\ a,\ a^3,\ a^5 = e$ ***A1***

successive powers of a^3 are $a^3,\ a,\ a^4,\ a^2,\ a^5 = e$ ***A1***

successive powers of a^4 are $a^4,\ a^3,\ a^2,\ a,\ a^5 = e$ ***A1***

this shows that $a^2,\ a^3,\ a^4$ are also generators in addition to a ***AG***

[4 marks]

Total [13 marks]

Examiners report

Solutions to (a) were often disappointing with some solutions even stating that a cyclic group is, by definition, commutative and therefore Abelian. Explanations in (b) were often poor and it was difficult in some cases to distinguish between correct and incorrect solutions. In (c), candidates who realised that Lagrange's Theorem could be used were generally the most successful. Solutions again confirmed that, in general, candidates find theoretical questions on this topic difficult.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 2\mathrm{e}^x - \mathrm{e}^{-x}.$$

(a) Show that f is a bijection.

(b) Find an expression for $f^{-1}(x)$.

Markscheme

(a) **EITHER**

consider

$f'(x) = 2\mathrm{e}^x - \mathrm{e}^{-x} > 0$ for all x ***M1A1***

so f is an injection ***A1***

OR

let $2\mathrm{e}^x - \mathrm{e}^{-x} = 2\mathrm{e}^y - \mathrm{e}^{-y}$ ***M1***

$2(\mathrm{e}^x - \mathrm{e}^y) + \mathrm{e}^{-y} - \mathrm{e}^{-x} = 0$

$2(\mathrm{e}^x - \mathrm{e}^y) + \mathrm{e}^{-(x+y)}(\mathrm{e}^x - \mathrm{e}^y) = 0$

$\left(2 + \mathrm{e}^{-(x+y)}\right)(\mathrm{e}^x - \mathrm{e}^y) = 0$

$\mathrm{e}^x = \mathrm{e}^y$

$x = y$ ***A1***

Note: Sufficient working must be shown to gain the above ***A1***.

so f is an injection ***A1***

Note: Accept a graphical justification *i.e.* horizontal line test.

THEN

it is also a surjection (accept any justification including graphical) ***R1***

therefore it is a bijection ***AG***

[4 marks]

(b) let $y = 2e^x - e^{-x}$ ***M1***

$2e^{2x} - ye^x - 1 = 0$ ***A1***

$e^x = \frac{y \pm \sqrt{y^2+8}}{4}$ ***M1A1***

since e^x is never negative, we take the + sign ***R1***

$f^{-1}(x) = \ln\left(\frac{x+\sqrt{x^2+8}}{4}\right)$ ***A1***

[6 marks]

Total [10 marks]

Examiners report

Solutions to (a) were often disappointing. Many candidates tried to use the result that, for an injection, $f(a) = f(b) \Rightarrow a = b$ – although this is the definition, it is often much easier to proceed by showing that the derivative is everywhere positive or everywhere negative or even to use a horizontal line test. Although (b) is based on core material, solutions were often disappointing with some very poor use of algebra seen.

The set of all permutations of the list of the integers 1, 2, 3 4 is a group, S_4, under the operation of function composition.

In the group S_4 let $p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$.

a. Determine the order of S_4. [2]

b. Find the proper subgroup H of order 6 containing p_1, p_2 and their compositions. Express each element of H in cycle form. [5]

c. Let $f: S_4 \to S_4$ be defined by $f(p) = p \circ p$ for $p \in S_4$. [5]

   Using p_1 and p_2, explain why f is not a homomorphism.

Markscheme

a. number of possible permutations is 4 × 3 × 2 × 1 ***(M1)***

   = 24(= 4!) ***A1***

   [2 marks]

b. attempting to find one of $p_1 \circ p_1$, $p_1 \circ p_2$ or $p_2 \circ p_1$ **M1**

   $p_1 \circ p_1 = (132)$ or equivalent (*eg*, ${p_1}^{-1} = (132)$) ***A1***

$p_1 \circ p_2 = (13)$ or equivalent (*eg*, $p_2 \circ p_1 \circ p_1 = (13)$) ***A1***

$p_2 \circ p_1 = (23)$ or equivalent (*eg*, $p_1 \circ p_1 \circ p_2 = (23)$) ***A1***

Note: Award ***A1A0A0*** for one correct permutation in any form; ***A1A1A0*** for two correct permutations in any form.

$e = (1), p_1 = (123)$ and $p_2 = (12)$ ***A1***

Note: Condone omission of identity in cycle form as long as it is clear it is considered one of the elements of *H*.

[5 marks]

c. **METHOD 1**

if f is a homomorphism $f(p_1 \circ p_2) = f(p_1) \circ f(p_2)$

attempting to express one of $f(p_1 \circ p_2)$ or $f(p_1) \circ f(p_2)$ in terms of p_1 and p_2 ***M1***

$f(p_1 \circ p_2) = p_1 \circ p_2 \circ p_1 \circ p_2$ ***A1***

$f(p_1) \circ f(p_2) = p_1 \circ p_1 \circ p_2 \circ p_2$ ***A1***

$\Rightarrow p_2 \circ p_1 = p_1 \circ p_2$ ***A1***

but $p_1 \circ p_2 \neq p_2 \circ p_1$ ***R1***

so f is not a homomorphism ***AG***

Note: Award ***R1*** only if ***M1*** is awarded.

Note: Award marks only if p_1 and p_2 are used; cycle form is not required.

METHOD 2

if f is a homomorphism $f(p_1 \circ p_2) = f(p_1) \circ f(p_2)$

attempting to find one of $f(p_1 \circ p_2)$ or $f(p_1) \circ f(p_2)$ ***M1***

$f(p_1 \circ p_2) = e$ ***A1***

$f(p_1) \circ f(p_2) = (132)$ ***(M1)A1***

so $f(p_1 \circ p_2) \neq f(p_1) \circ f(p_2)$ ***R1***

so f is not a homomorphism ***AG***

Note: Award ***R1*** only if ***M1*** is awarded.

Note: Award marks only if p_1 and p_2 are used; cycle form is not required.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

---

a. The relation aRb is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if and only if ab is the square of a positive integer. [10]

(i) Show that R is an equivalence relation.

(ii) Find the equivalence classes of R that contain more than one element.

b. Given the group $(G, *)$, a subgroup $(H, *)$ and $a, b \in G$, we define $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation. [9]

Markscheme

a. (i) $aRa \Rightarrow a \cdot a = a^2$ so R is reflexive ***A1***

$aRb = m^2 \Rightarrow bRa$ so R is symmetric ***A1***

$aRb = ab = m^2$ and $bRc = bc = n^2$ ***M1A1***

so $a = \frac{m^2}{b}$ and $c = \frac{n^2}{b}$

$ac = \frac{m^2n^2}{b^2} = \left(\frac{mn}{b}\right)^2,$ ***A1***

ac is an integer hence $\left(\frac{mn}{b}\right)^2$ is an integer ***R1***

so aRc, hence R is transitive ***R1***

R is therefore an equivalence relation ***AG***

(ii) $1R4$ and $4R9$ **or** $2R8$ ***M1***

so $\{1, 4, 9\}$ is an equivalence class ***A1***

and $\{2, 8\}$ is an equivalence class ***A1***

[10 marks]

b. $a \sim a$ since $aa^{-1} = e \in H$, the identity must be in H since it is a subgroup. ***M1***

Hence reflexivity. ***R1***

$a \sim b \Leftrightarrow ab^{-1} \in H$ but H is a subgroup so it must contain $(ab^{-1})^{-1} = ba^{-1}$ ***M1R1***

i.e. $ba^{-1} \in H$ so $\sim$ is symmetric ***A1***

$a \sim b$ and $b \sim c \Rightarrow ab^{-1} \in H$ and $bc^{-1} \in H$ ***M1***

But H is closed, so

$(ab^{-1})(bc^{-1}) \in H$ or $a(b^{-1}b)c^{-1} \in H$ ***R1***

$ac^{-1} \in H \Rightarrow a \sim c$ ***A1***

Hence $\sim$ is transitive and is thus an equivalence relation ***R1AG***

[9 marks]

Examiners report

a. Not a difficult question although using the relation definition to fully show transitivity was not well done. It was good to see some students use an operation binary matrix to show transitivity. This was a nice way given that the set was finite. The proof in (b) proved difficult.

b. Not a difficult question although using the relation definition to fully show transitivity was not well done. It was good to see some students use an operation binary matrix to show transitivity. This was a nice way given that the set was finite. The proof in (b) proved difficult.

Set $S = \{x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$ and a binary operation $\circ$ on S is defined as $x_i \circ x_j = x_k$, where $i + j \equiv k(\bmod 6)$.

(a) (i) Construct the Cayley table for $\{S,\ \circ\}$ and hence show that it is a group.

(ii) Show that $\{S,\ \circ\}$ is cyclic.

(b) Let $\{G,\ *\}$ be an Abelian group of order 6. The element $a \in$ G has order 2 and the element $b \in$ G has order 3.

(i) Write down the six elements of $\{G,\ *\}$.

(ii) Find the order of a $*$ b and hence show that $\{G,\ *\}$ is isomorphic to $\{S,\ \circ\}$.

Markscheme

(a) (i) Cayley table for $\{S,\ \circ\}$

$\circ$	x_0	x_1	x_2	x_3	x_4	x_5
x_0	x_0	x_1	x_2	x_3	x_4	x_5
x_1	x_1	x_2	x_3	x_4	x_5	x_0
x_2	x_2	x_3	x_4	x_5	x_0	x_1
x_3	x_3	x_4	x_5	x_0	x_1	x_2
x_4	x_4	x_5	x_0	x_1	x_2	x_3
x_5	x_5	x_0	x_1	x_2	x_3	x_4

A4

Note: Award ***A4*** for no errors, ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors and ***A0*** for four or more errors.

S is closed under $\circ$ ***A1***

x_0 is the identity ***A1***

x_0 and x_3 are self-inverses, ***A1***

x_2 and x_4 are mutual inverses and so are x_1 and x_5 ***A1***

modular addition is associative ***A1***

hence, $\{S,\ \circ\}$ is a group ***AG***

(ii) the order of x_1 (or x_5) is 6, hence there exists a generator, and $\{S,\ \circ\}$ is a cyclic group ***A1R1***

[11 marks]

(b) (i) *e*, *a*, *b*, *ab* ***A1***

and $b^2,\ ab^2$ ***A1A1***

Note: Accept ba and b^2a.

(ii) $(ab)^2 = b^2$ ***M1A1***

$(ab)^3 = a$ ***A1***

$(ab)^4 = b$ ***A1***

hence order is 6 ***A1***

groups *G* and *S* have the same orders and both are cyclic ***R1***

hence isomorphic ***AG***

[9 marks]

Total [20 marks]

Examiners report

a) Most candidates had the correct Cayley table and were able to show successfully that the group axioms were satisfied. Some candidates, however, simply stated that an inverse exists for each element without stating the elements and their inverses. Most candidates were able to find a generator and hence show that the group is cyclic.

b) This part was answered less successfully by many candidates. Some failed to find all the elements. Some stated that the order of *ab* is 6 without showing any working.

The function *f* is defined by

$$f(x) = \frac{1 - \mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}},\ x \in \mathbb{R}\,.$$

(a) Find the range of f.

(b) Prove that *f* is an injection.

(c) Taking the codomain of *f* to be equal to the range of f, find an expression for $f^{-1}(x)$.

Markscheme

(a) $]-1, 1[$ ***A1A1***

Note: Award ***A1*** for the values –1, 1 and ***A1*** for the open interval.

[2 marks]

(b) **EITHER**

Let $\frac{1-\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}} = \frac{1-\mathrm{e}^{-y}}{1+\mathrm{e}^{-y}}$ ***M1***

$1 - \mathrm{e}^{-x} + \mathrm{e}^{-y} - \mathrm{e}^{-(x+y)} = 1 + \mathrm{e}^{-x} - \mathrm{e}^{-y} - \mathrm{e}^{-(x+y)}$ ***A1***

$\mathrm{e}^{-x} = \mathrm{e}^{-y}$

$x = y$ ***A1***

Therefore *f* is an injection ***AG***

OR

Consider

$f'(x) = \frac{\mathrm{e}^{-x}(1+\mathrm{e}^{-x})+\mathrm{e}^{-x}(1-\mathrm{e}^{-x})}{(1+\mathrm{e}^{-x})^2}$ ***M1***

$= \frac{2\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}$ ***A1***

> 0 for all *x*. ***A1***

Therefore *f* is an injection. ***AG***

Note: Award ***M1A1A0*** for a graphical solution.

[3 marks]

(c) Let $y = \frac{1-\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}$ ***M1***

$y(1 + \mathrm{e}^{-x}) = 1 - \mathrm{e}^{-x}$ ***A1***

$\mathrm{e}^{-x}(1 + y) = 1 - y$ ***A1***

$\mathrm{e}^{-x} = \frac{1-y}{1+y}$

$x = \ln\left(\frac{1+y}{1-y}\right)$ ***A1***

$f^{-1}(x) = \ln\left(\frac{1+x}{1-x}\right)$ ***A1***

[5 marks]
Total [10 marks]

Examiners report

Most candidates found the range of f correctly. Two algebraic methods were seen for solving (b), either showing that the derivative of f is everywhere positive or showing that $f(a) = f(b) \Rightarrow a = b$. Candidates who based their 'proof' on a graph produced on their graphical calculators were given only partial credit on the grounds that the whole domain could not be shown and, in any case, it was not clear from the graph that f was an injection.

The relation R is defined on $\mathbb{Z} \times \mathbb{Z}$ such that $(a,\ b)R(c,\ d)$ if and only if $a - c$ is divisible by 3 and $b - d$ is divisible by 2.

(a) Prove that R is an equivalence relation.
(b) Find the equivalence class for (2, 1) .
(c) Write down the five remaining equivalence classes.

Markscheme

(a) consider $(x,\ y)R(x,\ y)$

since $x - x = 0$ and $y - y = 0$, R is reflexive ***A1***

assume $(x,\ y)R(a,\ b)$

$\Rightarrow x - a = 3M$ and $y - b = 2N$ ***M1***

$\Rightarrow a - x = -3M$ and $b - y = -2N$ ***A1***

$\Rightarrow (a,\ b)R(x,\ y)$

hence R is symmetric

assume $(x,\ y)R(a,\ b)$

$\Rightarrow x - a = 3M$ and $y - b = 2N$

assume $(a,\ b)R(c,\ d)$

$\Rightarrow a - c = 3P$ and $b - d = 2Q$ ***M1***

$\Rightarrow x - c = 3(M + P)$ and $y - d = 2(N + Q)$ ***A1***

hence $(x,\ y)R(c,\ d)$ ***A1***

hence R is transitive

therefore R is an equivalence relation ***AG***

[7 marks]

(b) $\{(x,\ y) : x = 3m + 2,\ y = 2n + 1,\ m,\ n \in \mathbb{Z}\}$ ***A1A1***

[2 marks]

(c) $\{3m,\ 2n\}$ $\{\ 3m + 1,\ 2n\}$ $\{3m + 2,\ 2n\}$

$\{3m,\ 2n+1\}\ \{\ 3m+1,\ 2n+1\}\ m,\ n \in \mathbb{Z}$ ***A1A1A1A1A1***

[5 marks]

Total [14 marks]

Examiners report

Stronger candidates had little problem with part (a) of this question, but proving an equivalence relation is still difficult for many. Equivalence classes still cause major problems and few fully correct answers were seen to this question.

The binary operation $*$ is defined on the set $S = \{0, 1, 2, 3\}$ by

$$a * b = a + 2b + ab(\text{ mod } 4)\ .$$

(a) (i) Construct the Cayley table.

(ii) Write down, with a reason, whether or not your table is a Latin square.

(b) (i) Write down, with a reason, whether or not $*$ is commutative.

(ii) Determine whether or not $*$ is associative, justifying your answer.

(c) Find all solutions to the equation $x * 1 = 2 * x$, for $x \in S$.

Markscheme

(a) (i)

	0	**1**	**2**	**3**
0	0	2	0	2
1	1	0	3	2
2	2	2	2	2
3	3	0	1	2

A3

Note: Award ***A3*** for no errors, ***A2*** for one error, ***A1*** for two errors and ***A0*** for three or more errors.

(ii) it is not a Latin square because some rows/columns contain the same digit more than once ***A1***

[4 marks]

(b) (i) **EITHER**

it is not commutative because the table is not symmetric about the leading diagonal ***R2***

OR

it is not commutative because $a + 2b + ab \neq 2a + b + ab$ in general ***R2***

Note: Accept a counter example *e.g.* $1 * 2 = 3$ whereas $2 * 1 = 2$.

(ii) **EITHER**

for example $(0 * 1) * 1 = 2 * 1 = 2$ ***M1***

and $0 * (1 * 1) = 0 * 0 = 0$ ***A1***

so $*$ is not associative ***A1***

OR

associative if and only if $a * (b * c) = (a * b) * c$ ***M1***

which gives

$a + 2b + 4c + 2bc + ab + 2ac + abc = a + 2b + ab + 2c + ac + 2bc + abc$ ***A1***

so $*$ is not associative as $2ac \neq 2c + ac$, in general ***A1***

[5 marks]

(c) $x = 0$ is a solution ***A2***

$x = 2$ is a solution ***A2***

[4 marks]

Total [13 marks]

Examiners report

This question was generally well answered.

(a) Find the six roots of the equation $z^6 - 1 = 0$, giving your answers in the form $r\,\text{cis}\,\theta, r \in \mathbb{R}^+, 0 \leqslant \theta < 2\pi$.

(b) (i) Show that these six roots form a group G under multiplication of complex numbers.

(ii) Show that G is cyclic and find all the generators.

(iii) Give an example of another group that is isomorphic to G, stating clearly the corresponding elements in the two groups.

Markscheme

(a) $z^6 = 1 = \text{cis}\,2n\pi$ ***(M1)***

The six roots are

$\text{cis}\,0(1),\ \text{cis}\frac{\pi}{3},\ \text{cis}\frac{2\pi}{3},\ \text{cis}\,\pi(-1),\ \text{cis}\frac{4\pi}{3},\ \text{cis}\frac{5\pi}{3}$ ***A3***

Note: Award ***A2*** for 4 or 5 correct roots, ***A1*** for 2 or 3 correct roots.

[4 marks]

(b) (i) Closure: Consider any two roots $\text{cis}\frac{m\pi}{3},\ \text{cis}\frac{n\pi}{3}$. ***M1***

$\text{cis}\frac{m\pi}{3} \times \text{cis}\frac{n\pi}{3} = \text{cis}\,(m+n)(\text{mod}6)\frac{\pi}{3} \in G$ ***A1***

Note: Award ***M1A1*** for a correct Cayley table showing closure.

Identity: The identity is 1. ***A1***

Inverse: The inverse of $\text{cis}\frac{m\pi}{3}$ is $\text{cis}\frac{(6-m)\pi}{3} \in G$. ***A2***

Associative: This follows from the associativity of multiplication. ***R1***

The 4 group axioms are satisfied. ***R1***

(ii) Successive powers of $\operatorname{cis}\frac{\pi}{3}\left(\text{or }\operatorname{cis}\frac{5\pi}{3}\right)$

generate the group which is therefore cyclic. ***R2***

The (only) other generator is $\operatorname{cis}\frac{5\pi}{3}\left(\text{or }\operatorname{cis}\frac{\pi}{3}\right)$. ***A1***

Note: Award ***A0*** for any additional answers.

(iii) The group of the integers 0, 1, 2, 3, 4, 5 under addition modulo 6. ***R2***

The correspondence is

$m \to \operatorname{cis}\frac{m\pi}{3}$ ***R1***

Note: Accept any other cyclic group of order 6.

[13 marks]

Total [17 marks]

Examiners report

This question was reasonably well answered by many candidates, although in (b)(iii), some candidates were unable to give another group isomorphic to G.

---

a. The relation R is defined on $\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied. [5]

b. The relation S is defined on $\mathbb{Z}^+$ by aSb if and only if $a^2 \equiv b^2(\bmod 6)$. [9]

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members.

Markscheme

a. reflexive: if a is odd, $a \times a$ is odd so R is not reflexive ***R1***

symmetric: if ab is even then ba is even so R is symmetric ***R1***

transitive: let aRb and bRc; it is necessary to determine whether or not aRc ***(M1)***

for example $5R2$ and $2R3$ ***A1***

since 5×3 is not even, 5 is not related to 3 and R is not transitive ***R1***

[5 marks]

b. (i) reflexive: $a^2 \equiv a^2(\bmod 6)$ so S is reflexive ***R1***

symmetric: $a^2 \equiv b^2(\bmod 6) \Rightarrow 6|(a^2 - b^2) \Rightarrow 6|(b^2 - a^2) \Rightarrow b^2 \equiv a^2(\bmod 6)$ ***R1***

so S is symmetric

transitive: let aSb and bSc so that $a^2 = b^2 + 6M$ and $b^2 = c^2 + 6N$ ***M1***

it follows that $a^2 = c^2 + 6(M + N)$ so aSc and S is transitive ***R1***

S is an equivalence relation because it satisfies the three conditions ***AG***

(ii) by considering the squares of integers (mod 6), the equivalence ***(M1)***

classes are

$\{1, 5, 7, 11, \ldots\}$ ***A1***

$\{2, 4, 8, 10, \ldots\}$ ***A1***

$\{3, 9, 15, 21, \ldots\}$ ***A1***

$\{6, 12, 18, 24, \ldots\}$ ***A1***

[9 marks]

Examiners report

a. [N/A]
b. [N/A]

The groups $\{K,\ *\}$ and $\{H,\ \odot\}$ are defined by the following Cayley tables.

G

$*$	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	A	E
C	C	B	E	A

H

$\odot$	e	a
e	e	a
a	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

Markscheme

consider the function f given by

$f(E) = e$

$f(A) = e$

$f(B) = a$ ***M1A1***

$f(C) = a$

then, it has to be shown that

$f(X * Y) = f(X) \odot f(Y)$ for all $X, Y \in G$ ***(M1)***

consider

$f\left((E \text{ or } A) * (E \text{ or } A)\right) = f(E \text{ or } A) = e;\ f(E \text{ or } A) \odot f(E \text{ or } A) = e \odot e = e$ ***M1A1***

$f\left((E \text{ or } A) * (B \text{ or } C)\right) = f(B \text{ or } C) = a;\ f(E \text{ or } A) \odot f(B \text{ or } C) = e \odot a = a$ ***A1***

$f\left((B \text{ or } C) * (B \text{ or } C)\right) = f(E \text{ or } A) = e;\ f(B \text{ or } C) \odot f(B \text{ or } C) = a \odot a = e$ ***A1*** since the groups are Abelian, there is no need to consider $f\left((B \text{ or } C) * (E \text{ or } A)\right)$ ***R1***

the required property is satisfied in all cases so the homomorphism exists

Note: A comprehensive proof using tables is acceptable.

the kernel is $\{E,\ A\}$ ***A1***

[9 marks]

Examiners report

[N/A]

Three functions mapping $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ are defined by

$$f_1(m,\ n) = m - n + 4;\ \ f_2(m,\ n) = |m|\,;\ \ f_3(m,\ n) = m^2 - n^2.$$

Two functions mapping $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ are defined by

$$g_1(k) = (2k,\ k);\ \ g_2(k) = (k,\ |k|)\,.$$

(a) Find the range of

(i) $f_1 \circ g_1$;

(ii) $f_3 \circ g_2$.

(b) Find all the solutions of $f_1 \circ g_2(k) = f_2 \circ g_1(k)$.

(c) Find all the solutions of $f_3(m,\ n) = p$ in each of the cases $p = 1$ and $p = 2$.

Markscheme

(a) (i) $f_1 \circ g_1(k) = k + 4$ ***M1***

Range $(f_1 \circ g_1) = \mathbb{Z}$ ***A1***

(ii) $f_3 \circ g_2(k) = 0$ ***M1***

Range $(f_3 \circ g_2) = \{0\}$ ***A1***

[4 marks]

(b) the equation to solve is

$k - |k| + 4 = |2k|$ ***M1A1***

the positive solution is $k = 2$ ***A1***

the negative solution is $k = -1$ ***A1***

[4 marks]

(c) the equation factorizes: $(m+n)(m-n)=p$ ***(M1)***

for $p = 1$, the possible factors over $\mathbb{Z}$ are $m+n=\pm 1,\ m-n=\pm 1$ ***(M1)(A1)***

with solutions (1, 0) and (–1, 0) ***A1***

for $p = 2$, the possible factors over $\mathbb{Z}$ are $m+n=\pm 1,\ \pm 2;\ m-n=\pm 2,\ \pm 1$ ***M1A1***

there are no solutions over $\mathbb{Z}\times\mathbb{Z}$ ***A1***

[7 marks]

Total [15 marks]

Examiners report

The majority of candidates were able to compute the composite functions involved in parts (a) and (b). Part(c) was satisfactorily tackled by a minority of candidates. There were more GDC solutions than the more obvious approach of factorizing a difference of squares. Some candidates seemed to forget that *m* and *n* belonged to the set of integers.

$\{G,\ *\}$ is a group with identity element e. Let $a,\ b\in G$.

a. State Lagrange's theorem. [2]

b. Verify that the inverse of $a*b^{-1}$ is equal to $b*a^{-1}$. [3]

c. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$. Let R be a relation defined on G by [8]

$$aRb\Leftrightarrow a*b^{-1}\in H.$$

Prove that R is an equivalence relation, indicating clearly whenever you are using one of the four properties required of a group.

d. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$ .Let R be a relation defined on G by [3]

$$aRb\Leftrightarrow a*b^{-1}\in H.$$

Show that $aRb\Leftrightarrow a\in Hb$, where Hb is the right coset of H containing b.

e. Let $\{H,\ *\}$ be a subgroup of $\{G,\ *\}$ .Let R be a relation defined on G by [3]

$$aRb\Leftrightarrow a*b^{-1}\in H.$$

It is given that the number of elements in any right coset of H is equal to the order of H.

Explain how this fact together with parts (c) and (d) prove Lagrange's theorem.

Markscheme

a. in a **finite** group the order of any subgroup (exactly) divides the order of the group ***A1A1***

[2 marks]

b. **METHOD 1**

$(a * b^{-1}) * (b * a^{-1}) = a * b^{-1} * b * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$ ***M1A1A1***

Note: ***M1*** for multiplying, ***A1*** for at least one of the next 3 expressions,

A1 for e.

Allow $(b * a^{-1}) * (a * b^{-1}) = b * a^{-1} * a * b^{-1} = b * e * b^{-1} = b * b^{-1} = e$.

METHOD 2

$(a * b^{-1})^{-1} = (b^{-1})^{-1} * a^{-1}$ ***M1A1***

$= b * a^{-1}$***A1***

[3 marks]

c. $a * a^{-1} = e \in H$ (as H is a subgroup) ***M1***

so aRa and hence R is reflexive

$aRb \Leftrightarrow a * b^{-1} \in H$. H is a subgroup so every element has an inverse in H so

$(a * b^{-1})^{-1} \in H$ ***R1***

$\Leftrightarrow b * a^{-1} \in H \Leftrightarrow bRa$ ***M1***

so R is symmetric

$aRb,\ bRc \Leftrightarrow a * b^{-1} \in H,\ b * c^{-1} \in H$ ***M1***

as H is closed $(a * b^{-1}) * (b * c^{-1}) \in H$ ***R1***

and using associativity ***R1***

$(a * b^{-1}) * (b * c^{-1}) = a * (b^{-1} * b) * c^{-1} = a * c^{-1} \in H \Leftrightarrow aRc$ ***A1***

therefore R is transitive

R is reflexive, symmetric and transitive

Note: Can be said separately at the end of each part.

hence it is an equivalence relation ***AG***

[8 marks]

d. $aRb \Leftrightarrow a * b^{-1} \in H \Leftrightarrow a * b^{-1} = h \in H$ ***A1***

$\Leftrightarrow a = h * b \Leftrightarrow a \in Hb$ ***M1R1***

[3 marks]

e. (d) implies that the right cosets of H are equal to the equivalence classes of the relation in (c) ***R1***

hence the cosets partition G ***R1***

all the cosets are of the same size as the subgroup H so the order of G must be a multiple of $|H|$ ***R1***

[3 marks]

Total [19 marks]

Examiners report

a. Many students obtained just half marks in (a) for not stating the requirement of the order to be finite.

b. Part (b) should have been more straightforward than many found.

c. In part (c) it was evident that most candidates knew what to do, but being a more difficult question fell down on a lack of rigour. Nonetheless, many candidates obtained full or partial marks on this question part.

d. Part (d) enabled many candidates to obtain, at least partial marks, but there were few students with the insight to be able to answer part (e) satisfactorily.

e. Part (d) enabled many candidates to obtain, at least partial marks, but there were few students with the insight to be able to answer part (e) satisfactorily.

(a) Given a set U, and two of its subsets A and B, prove that

$$(A\backslash B) \cup (B\backslash A) = (A \cup B)\backslash(A \cap B), \text{ where } A\backslash B = A \cap B'.$$

(b) Let $S = \{A,\ B,\ C,\ D\}$ where $A = \emptyset,\ B = \{0\},\ C = \{0,\ 1\}$ and $D = \{0, 1, 2\}$.
State, with reasons, whether or not each of the following statements is true.
(i) The operation \ is closed in S.
(ii) The operation $\cap$ has an identity element in S but not all elements have an inverse.
(iii) Given $Y \in S$, the equation $X \cup Y = Y$ always has a unique solution for X in S.

Markscheme

(a) $(A\backslash B) \cup (B\backslash A) = (A \cap B') \cup (B \cap A')$ ***(M1)***

$= ((A \cap B') \cup B) \cap ((A \cap B') \cup A')$ ***M1***

$= \left((A \cup B) \cap \underbrace{(B' \cup B)}_{U}\right) \cap \left(\underbrace{(A \cup A')}_{U} \cap (B' \cup A')\right)$ ***A1***

$= (A \cup B') \cap (B' \cup A') = (A \cup B) \cap (A \cap B)'$ ***A1***

$= (A \cup B)\backslash(A \cap B)$ ***AG***

[4 marks]

(b) (i) false ***A1***

counterexample

eg $D\backslash C = \{2\} \notin S$ ***R1***

(ii) true ***A1***

as $A \cap D = A,\ B \cap D = B,\ C \cap D = C$ and $D \cap D = D$,

D is the identity ***R1***

A (or B or C) has no inverse as $A \cap X = D$ is impossible ***R1***

(iii) false ***A1***

when $Y = D$ the equation has more than one solution (four solutions) ***R1***

[7 marks]

Examiners report

For part (a), candidates who chose to prove the given statement using the properties of Sets were often successful with the proof. Some candidates chose to use the definition of equality of sets, but made little to no progress. In a few cases candidates attempted to use Venn diagrams as a proof. Part (b) was challenging for most candidates, and few correct answers were seen.

The relation R is defined on $\mathbb{Z}$ by xRy if and only if $x^2y \equiv y \bmod 6$.

a. Show that the product of three consecutive integers is divisible by 6. [2]

b. Hence prove that R is reflexive. [3]

c. Find the set of all y for which $5Ry$. [3]

d. Find the set of all y for which $3Ry$. [2]

e. Using your answers for (c) and (d) show that R is not symmetric. [2]

Markscheme

a. in a product of three consecutive integers either one or two are even ***R1***

and one is a multiple of 3 ***R1***

so the product is divisible by 6 ***AG***

[2 marks]

b. to test reflexivity, put $y = x$ ***M1***

then $x^2x - x = (x-1)x(x+1) \equiv 0 \bmod 6$ ***M1A1***

so xRx ***AG***

[3 marks]

c. if $5Ry$ then $25y \equiv y \bmod 6$ ***(M1)***

$24y \equiv 0 \bmod 6$ ***(M1)***

the set of solutions is $\mathbb{Z}$ ***A1***

Note: Only one of the method marks may be implied.

[3 marks]

d. if $3Ry$ then $9y \equiv y \bmod 6$

$8y \equiv 0 \bmod 6 \Rightarrow 4y \equiv 0 \bmod 3$ ***(M1)***

the set of solutions is $3\mathbb{Z}$ (*ie* multiples of 3) ***A1***

[2 marks]

e. from part (c) $5R3$ ***A1***

from part (d) $3R5$ is false ***A1***

R is not symmetric ***AG***

Note: Accept other counterexamples.

[2 marks]

Total [12 marks]

Examiners report

a. A surprising number of candidates thought that an example was sufficient evidence to answer this part.

b. Again, a lack of confidence with modular arithmetic undermined many candidates' attempts at this part.

c. (c) and (d) Most candidates started these parts, but some found solutions as fractions rather than integers or omitted zero and/or negative integers.

d. (c) and (d) Most candidates started these parts, but some found solutions as fractions rather than integers or omitted zero and/or negative integers.

e. Some candidates regarded R as an operation, rather than a relation, so returned answers of the form $aRb \neq bRa$.

Determine, giving reasons, which of the following sets form groups under the operations given below. Where appropriate you may assume that multiplication is associative.

(a) $\mathbb{Z}$ under subtraction.

(b) The set of complex numbers of modulus 1 under multiplication.

(c) The set $\{1, 2, 4, 6, 8\}$ under multiplication modulo 10.

(d) The set of rational numbers of the form

$$\frac{3m+1}{3n+1}, \text{ where } m,\ n \in \mathbb{Z}$$

under multiplication.

Markscheme

(a) not a group ***A1***

EITHER

subtraction is not associative on $\mathbb{Z}$ (or give counter-example) ***R1***

OR

there is a right-identity, 0, but it is not a left-identity ***R1***

[2 marks]

(b) the set forms a group ***A1***

the closure is a consequence of the following relation (and the closure of $\mathbb{C}$ itself):

$|z_1 z_2| = |z_1| \, |z_2|$ ***R1***

the set contains the identity 1 ***R1***

that inverses exist follows from the relation

$|z^{-1}| = |z|^{-1}$

for non-zero complex numbers ***R1***

[4 marks]

(c) not a group ***A1***

for example, only the identity element 1 has an inverse ***R1***

[2 marks]

(d) the set forms a group ***A1***

$\frac{2m+1}{3n+1} \times \frac{3s+1}{3t+1} = \frac{9ms+3s+3m+1}{9nt+3n+3t+1} = \frac{3(3ms+s+m)+1}{3(3nt+n+t)+1}$ ***M1R1***

shows closure

the identity 1 corresponds to *m* = *n* = 0 ***R1***

an inverse corresponds to interchanging the parameters *m* and *n* ***R1***

[5 marks]

Total [13 marks]

Examiners report

There was a mixed response to this question. Some candidates were completely out of their depth. Stronger candidates provided satisfactory answers to parts (a) and (c). For the other parts there was a general lack of appreciation that, for example, closure and the existence of inverses, requires that products and inverses have to be shown to be members of the set.

Consider the set *S* defined by $S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.

You may assume that $+$ (addition) and $\times$ (multiplication) are associative binary operations on $\mathbb{Q}$.

a. (i) Write down the six smallest non-negative elements of S. [9]

(ii) Show that $\{S, +\}$ is a group.

(iii) Give a reason why $\{S, \times\}$ is not a group. Justify your answer.

b. The relation R is defined on S by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$. [10]

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence classes.

Markscheme

a. (i) $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ ***A2***

Notes: ***A2*** for all correct, ***A1*** for three to five correct.

(ii) **EITHER**
closure: if $s_1,\ s_2 \in S$, then $s_1 = \frac{m}{2}$ and $s_2 = \frac{n}{2}$ for some $m,\ n \in ¢$. ***M1***

Note: Accept two distinct examples (*eg*, $\frac{1}{2} + \frac{1}{2} = 1;\ \frac{1}{2} + 1 = \frac{3}{2}$) for the ***M1***.

$s_1 + s_2 = \frac{m+n}{2} \in S$ ***A1***
OR
the sum of two half-integers ***A1***
is a half-integer ***R1***
THEN
identity: 0 is the (additive) identity ***A1***
inverse: $s + (-s) = 0$, where $-s \in S$ ***A1***
it is associative (since $S \subset §$) ***A1***
the group axioms are satisfied ***AG***
(iii) **EITHER**
the set is not closed under multiplication, ***A1***
for example, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, but $\frac{1}{4} \notin S$ ***R1***
OR
not every element has an inverse, ***A1***
for example, 3 does not have an inverse ***R1***
[9 marks]

b. (i) reflexive: consider $3s + 5s$ ***M1***

$= 8s \in ¢ \Rightarrow$ reflexive ***A1***
symmetric: if $s_1 R s_2$, consider $3s_2 + 5s_1$ ***M1***
for example, $= 3s_1 + 5s_2 + (2s_1 - 2s_2) \in ¢ \Rightarrow$symmetric ***A1***
transitive: if $s_1 R s_2$ and $s_2 R s_3$, consider ***(M1)***
$3s_1 + 5s_3 = (3s_1 + 5s_2) + (3s_2 + 5s_3) - 8s_2$ ***M1***
$\in ¢ \Rightarrow$transitive ***A1***
so R is an equivalence relation ***AG***
(ii) $C_1 = ¢$ ***A1***
$C_2 = \left\{\pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \ldots\right\}$ ***A1A1***

Note: ***A1*** for half odd integers and ***A1*** for $\pm$.

[10 marks]

Examiners report

a. [N/A]
b. [N/A]

The binary operation Δ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

Δ	1	2	3	4	5
1	1	1	2	3	4
2	1	2	1	2	3
3	2	1	3	1	2
4	3	2	1	4	1
5	4	3	2	1	5

(a) State whether S is closed under the operation Δ and justify your answer.

(b) State whether Δ is commutative and justify your answer.

(c) State whether there is an identity element and justify your answer.

(d) Determine whether Δ is associative and justify your answer.

(e) Find the solutions of the equation $a\Delta b = 4\Delta b$, for $a \neq 4$.

Markscheme

(a) yes ***A1***

because the Cayley table only contains elements of S ***R1***
[2 marks]

(b) yes ***A1***
because the Cayley table is symmetric ***R1***
[2 marks]

(c) no ***A1***
because there is no row (and column) with 1, 2, 3, 4, 5 ***R1***
[2 marks]

(d) attempt to calculate $(a\Delta b)\Delta c$ and $a\Delta(b\Delta c)$ for some $a,\ b,\ c \in S$ ***M1***
counterexample: for example, $(1\Delta 2)\Delta 3 = 2$
$1\Delta(2\Delta 3) = 1$ ***A1***
Δ is not associative ***A1***

Note: Accept a correct evaluation of $(a\Delta b)\Delta c$ and $a\Delta(b\Delta c)$ for some $a,\ b,\ c \in S$ for the ***M1***.

[3 marks]

(e) for example, attempt to enumerate $4\Delta b$ for b = 1, 2, 3, 4, 5 and obtain (3, 2, 1, 4, 1) ***(M1)***
find $(a,\ b) \in \{(2,\ 2),\ (2,\ 3)\}$ for $a \neq 4$ (or equivalent) ***A1A1***

Note: Award ***M1A1A0*** if extra 'solutions' are listed.

[3 marks]

Total [12 marks]

Examiners report

[N/A]

The binary operation $*$ is defined for $a, b \in \mathbb{Z}^+$ by

$$a * b = a + b - 2.$$

(a) Determine whether or not $*$ is

(i) closed,

(ii) commutative,

(iii) associative.

(b) (i) Find the identity element.

(ii) Find the set of positive integers having an inverse under $*$.

Markscheme

(a) (i) It is not closed because

$1 * 1 = 0 \notin \mathbb{Z}^+$. ***R2***

(ii) $a * b = a + b - 2$
$b * a = b + a - 2 = a * b$ ***M1***
It is commutative. ***A1***

(iii) It is not associative. ***A1***
Consider $(1 * 1) * 5$ and $1 * (1 * 5)$.
The first is undefined because $1 * 1 \notin \mathbb{Z}^+$.
The second equals 3. ***R2***
Notes: Award ***A1R2*** for stating that non-closure implies non-associative.
Award ***A1R1*** to candidates who show that $a * (b * c) = (a * b) * c = a + b + c - 4$ and therefore conclude that it is associative, ignoring the non-closure.
[7 marks]

(b) (i) The identity e satisfies
$a * e = a + e - 2 = a$ ***M1***
$e = 2 \quad (\text{and } 2 \in \mathbb{Z}^+)$ ***A1***

(ii) $a * a^{-1} = a + a^{-1} - 2 = 2$ ***M1***
$a + a^{-1} = 4$ ***A1***
So the only elements having an inverse are 1, 2 and 3. ***A1***
Note: Due to commutativity there is no need to check two sidedness of identity and inverse.

[5 marks]
Total [12 marks]

Examiners report

Almost all the candidates thought that the binary operation was associative, not realising that the non-closure prevented this from being the case.

In the circumstances, however, partial credit was given to candidates who ‘proved’ associativity. Part (b) was well done by many candidates.

A, B, C and D are subsets of $\mathbb{Z}$.

$$A = \{m | m \text{ is a prime number less than } 15\}$$
$$B = \{m | m^4 = 8m\}$$

$C = \{m|\,(m+1)(m-2) < 0\}$

$D = \{m|\,m^2 < 2m+4\}$

(a) List the elements of each of these sets.

(b) Determine, giving reasons, which of the following statements are true and which are false.

(i) $n(D) = n(B) + n(B \cup C)$

(ii) $D \backslash B \subset A$

(iii) $B \cap A' = \emptyset$

(iv) $n(B \Delta C) = 2$

Markscheme

(a) by inspection, or otherwise,

$A = \{2, 3, 5, 7, 11, 13\}$ ***A1***

$B = \{0, 2\}$ ***A1***

$C = \{0, 1\}$ ***A1***

$D = \{-1, 0, 1, 2, 3\}$ ***A1***

[4 marks]

(b) (i) true ***A1***

$n(B) + n(B \cup C) = 2 + 3 = 5 = n(D)$ ***R1***

(ii) false ***A1***

$D \backslash B = \{-1,\ 1,\ 3\} \not\subset A$ ***R1***

(iii) false ***A1***

$B \cap A' = \{0\} \neq \emptyset$ ***R1***

(iv) true ***A1***

$n(B \Delta C) = n\{1,\ 2\} = 2$ ***R1***

[8 marks]

Total [12 marks]

Examiners report

It was surprising and disappointing that many candidates regarded 1 as a prime number. One of the consequences of this error was that it simplified some of the set-theoretic calculations in part(b), with a loss of follow-through marks. Generally speaking, it was clear that the majority of candidates were familiar with the set operations in part(b).

(a) Consider the set $A = \{1, 3, 5, 7\}$ under the binary operation $*$, where $*$ denotes multiplication modulo 8.

(i) Write down the Cayley table for $\{A,\ *\}$.

(ii) Show that $\{A,\ *\}$ is a group.

(iii) Find all solutions to the equation $3 * x * 7 = y$. Give your answers in the form $(x,\ y)$.

(b) Now consider the set $B = \{1, 3, 5, 7, 9\}$ under the binary operation $\otimes$, where $\otimes$ denotes multiplication modulo 10. Show that $\{B,\ \otimes\}$ is not a group.

(c) Another set C can be formed by removing an element from B so that $\{C,\ \otimes\}$ is a group.

(i) State which element has to be removed.

(ii) Determine whether or not $\{A,\ *\}$ and $\{C,\ \otimes\}$ are isomorphic.

Markscheme

(a) (i)

$*$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

A3

Note: Award ***A2*** for 15 correct, ***A1*** for 14 correct and ***A0*** otherwise.

(ii) it is a group because:

the table shows closure ***A1***

multiplication is associative ***A1***

it possesses an identity 1 ***A1***

justifying that every element has an inverse *e.g.* all self-inverse ***A1***

(iii) (since $*$ is commutative, $5 * x = y$)

so solutions are (1, 5), (3, 7), (5, 1), (7, 3) ***A2***

Notes: Award ***A1*** for 3 correct and ***A0*** otherwise.

Do not penalize extra incorrect solutions.

[9 marks]

(b)

$\otimes$	1	3	5	7	9
1	1	3	5	7	9
3	3	9	5	1	7
5	5	5	5	5	5
7	7	1	5	9	3
9	9	7	5	3	1

Note: It is not necessary to see the Cayley table.

a valid reason ***R2***

e.g. from the Cayley table the 5 row does not give a Latin square, or 5 does not have an inverse, so it cannot be a group

[2 marks]

(c) (i) remove the 5 ***A1***

(ii) they are not isomorphic because all elements in A are self-inverse this is not the case in C, (e.g. $3 \otimes 3 = 9 \neq 1$) ***R2***

Note: Accept any valid reason.

[3 marks]

Total [14 marks]

Examiners report

Candidates are generally confident when dealing with a specific group and that was the situation again this year. Some candidates lost marks in (a)(ii) by not giving an adequate explanation for the truth of some of the group axioms, eg some wrote ‘every element has an inverse’. Since the question told the candidates that $\{A, *)$ was a group, this had to be the case and the candidates were expected to justify their statement by noting that every element was self-inverse. Solutions to (c)(ii) were reasonably good in general, certainly better than solutions to questions involving isomorphisms set in previous years.

Let $\{G, *\}$ be a finite group of order n and let H be a non-empty subset of G .

(a) Show that any element $h \in H$ has order smaller than or equal to n .

(b) If H is closed under $*$, show that $\{H, *\}$ is a subgroup of $\{G, *\}$.

Markscheme

(a) if $h \in H$ then $h \in G$ ***R1***

hence, (by Lagrange) the order of h exactly divides n

and so the order of h is smaller than or equal to n ***R2***

[3 marks]

(b) the associativity in G ensures associativity in H ***R1***

(closure within H is given)

as H is non-empty there exists an $h \in H$, let the order of h be m then $h^m = e$ and as H is closed $e \in H$ ***R2***

it follows from the earlier result that $h * h^{m-1} = h^{m-1} * h = e$ ***R1***

thus, the inverse of h is h^{m-1} which $\in H$ ***R1***

the four axioms are satisfied showing that $\{H, *\}$ is a subgroup ***R1***

[6 marks]

Total [9 marks]

Examiners report

Solutions to this question were extremely disappointing. This property of subgroups is mentioned specifically in the Guide and yet most candidates were unable to make much progress in (b) and even solutions to (a) were often unconvincing.

The group $\{G, *\}$ is Abelian and the bijection $f: G \to G$ is defined by $f(x) = x^{-1},\ x \in G$.

Show that f is an isomorphism.

Markscheme

we need to show that $f(a * b) = f(a) * f(b)$ ***R1***

Note: This ***R1*** may be awarded at any stage.

let $a,\ b \in G$ ***(M1)***

consider $f(a) * f(b)$ ***M1***

$= a^{-1} * b^{-1}$ ***A1***

consider $f(a * b) = (a * b)^{-1}$ ***M1***

$= b^{-1} * a^{-1}$ ***A1***

$= a^{-1} * b^{-1}$ since G is Abelian ***R1***

hence f is an isomorphism ***AG***

[7 marks]

Examiners report

A surprising number of candidates wasted time and unrewarded effort showing that the mapping f, stated to be a bijection in the question, actually was a bijection. Many candidates failed to get full marks by not properly using the fact that the group was stated to be Abelian. There were also candidates who drew the graph of $y = \frac{1}{x}$ or otherwise assumed that the inverse of x was its reciprocal - this is unacceptable in the context of an abstract group question.

The group G has a subgroup H. The relation R is defined on G by xRy if and only if $xy^{-1} \in H$, for $x,\ y \in G$.

a. Show that R is an equivalence relation. [8]

b. The Cayley table for G is shown below. [6]

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

The subgroup H is given as $H = \{e,\ a^2b\}$.

(i) Find the equivalence class with respect to R which contains ab.

(ii) Another equivalence relation ρ is defined on G by $x\rho y$ if and only if $x^{-1}y \in H$, for $x,\ y \in G$. Find the equivalence class with respect to ρ which contains ab.

Markscheme

a. $xx^{-1} = e \in H$ ***M1***

$\Rightarrow xRx$

hence R is reflexive ***A1***

if xRy then $xy^{-1} \in H$

$\Rightarrow (xy^{-1})^{-1} \in H$ ***M1***

now $(xy^{-1})(xy^{-1})^{-1} = e$ and $xy^{-1}yx^{-1} = e$

$\Rightarrow (xy^{-1})^{-1} = yx^{-1}$ ***A1***

hence $yx^{-1} \in H \Rightarrow yRx$

hence R is symmetric ***A1***

if xRy, yRz then $xy^{-1} \in H,\ yz^{-1} \in H$ ***M1***

$\Rightarrow (xy^{-1})(yz^{-1}) \in H$ ***M1***

$\Rightarrow x(y^{-1}y)z^{-1} \in H$

$\Rightarrow x^{-1}z \in H$

hence R is transitive ***A1***

hence R is an equivalence relation ***AG***

[8 marks]

b. (i) for the equivalence class, solving:

EITHER

$x(ab)^{-1} = e$ or $x(ab)^{-1} = a^2b$ ***(M1)***

$\{ab,\ a\}$ ***A2***

OR

$ab(x)^{-1} = e$ or $ab(x)^{-1} = a^2b$ ***(M1)***

$\{ab,\ a\}$ ***A2***

(ii) for the equivalence class, solving:

EITHER

$x^{-1}(ab) = e$ or $x^{-1}(ab) = a^2b$ ***(M1)***

$\{ab,\ a^2\}$ ***A2***

OR

$(ab)^{-1}x = e$ or $(ab)^{-1}x = a^2b$ ***(M1)***

$\{ab,\ a^2\}$ ***A2***

[6 marks]

Examiners report

a. Stronger candidates made a reasonable start to (a), and many were able to demonstrate that the relation was reflexive and transitive. However, the majority of candidates struggled to make a meaningful attempt to show the relation was symmetric, with many making unfounded assumptions. Equivalence classes still cause major problems and few fully correct answers were seen to (b).

b. Stronger candidates made a reasonable start to (a), and many were able to demonstrate that the relation was reflexive and transitive. However, the majority of candidates struggled to make a meaningful attempt to show the relation was symmetric, with many making unfounded assumptions. Equivalence classes still cause major problems and few fully correct answers were seen to (b).

The relation R is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by aRb if and only if $a(a+1) \equiv b(b+1)(\bmod 5)$.

a. Show that R is an equivalence relation. [6]

b. Show that the equivalence defining R can be written in the form [3]

$$(a-b)(a+b+1) \equiv 0(\bmod 5).$$

c. Hence, or otherwise, determine the equivalence classes. [4]

Markscheme

a. reflexive: $a(a+1) \equiv a(a+1)(\bmod 5)$, therefore aRa ***R1***

symmetric: $aRb \Rightarrow a(a+1) = b(b+1) + 5N$ ***M1***

$\Rightarrow b(b+1) = a(a+1) - 5N \Rightarrow bRa$ ***A1***

transitive:

EITHER

aRb and $bRc \Rightarrow a(a+1) = b(b+1) + 5M$ and $b(b+1) = c(c+1) + 5N$ ***M1***

it follows that $a(a+1) = c(c+1) + 5(M+N) \Rightarrow aRc$ ***M1A1***

OR

aRb and $bRc \Rightarrow a(a+1) \equiv b(b+1)(\bmod 5)$ and

$b(b+1) \equiv c(c+1)(\bmod 5)$ ***M1***

$a(a+1) - b(b+1) \equiv 0(\bmod 5);\ b(b+1) - c(c+1) \equiv 0(\bmod 5)$ ***M1***

$a(a+1) - c(c+1) \equiv 0 \bmod 5 \Rightarrow a(a+1) \equiv c(c+1) \bmod 5 \Rightarrow aRc$ ***A1***

[6 marks]

b. the equivalence can be written as

$a^2 + a - b^2 - b \equiv 0(\bmod 5)$ ***M1***

$(a-b)(a+b) + a - b \equiv 0(\bmod 5)$ ***M1A1***

$(a-b)(a+b+1) \equiv 0(\bmod 5)$ ***AG***

[3 marks]

c. the equivalence classes are

$\{1, 3, 6, 8, 11\}$

$\{2, 7, 12\}$

$\{4, 5, 9, 10\}$ ***A4***

Note: Award ***A3*** for 2 correct classes, ***A2*** for 1 correct class.

[4 marks]

Examiners report

a. Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

b. Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

c. Candidates knew the properties of equivalence relations but did not show sufficient working out in the transitive case. Others did not do the modular arithmetic correctly, still others omitted the $\bmod (5)$ in part or throughout.

Consider the set $S_3 = \{p, q, r, s, t, u\}$ of permutations of the elements of the set $\{1, 2, 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},\ s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $\circ$ denote composition of permutations, so $a \circ b$ means b followed by a. You may assume that $(S_3, \circ)$ forms a group.

a. Complete the following Cayley table [4]

$\circ$	p	q	r	s	t	u
p						
q			t			s
r		u		t	s	q
s		t	u			r
t		s	q	r		
u		r	s	q		

[5 marks]

b. (i) State the inverse of each element. [6]

(ii) Determine the order of each element.

c. Write down the subgroups containing [2]

(i) r,

(ii) u.

Markscheme

a.

$\circ$	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	p	t	u	r	s
r	r	u	p	t	s	q
s	s	t	u	p	q	r
t	t	s	q	r	u	p
u	u	r	s	q	p	t

(M1)A4

Note: Award ***M1*** for use of Latin square property and/or attempted multiplication, ***A1*** for the first row or column, ***A1*** for the squares of q, r and s, then ***A2*** for all correct.

b. (i) $p^{-1} = p,\ q^{-1} = q,\ r^{-1} = r,\ s^{-1} = s$ ***A1***

$t^{-1} = u,\ u^{-1} = t$ ***A1***

Note: Allow FT from part (a) unless the working becomes simpler.

(ii) using the table or direct multiplication ***(M1)***

the orders of $\{p,\ q,\ r,\ s,\ t,\ u\}$ are $\{1,\ 2,\ 2,\ 2,\ 3,\ 3\}$ ***A3***

Note: Award ***A1*** for two, three or four correct, ***A2*** for five correct.

[6 marks]

c. (i) $\{p,\ r\}\ (\text{and } (S_3,\ \circ))$ ***A1***

(ii) $\{p,\ u,\ t\}\ (\text{and } (S_3,\ \circ))$ ***A1***

Note: Award ***A0A1*** if the identity has been omitted.

Award ***A0*** in (i) or (ii) if an extra incorrect "subgroup" has been included.

[2 marks]

Total [13 marks]

Examiners report

a. The majority of candidates were able to complete the Cayley table correctly. Unfortunately, many wasted time and space, laboriously working out the missing entries in the table - the identity is p and the elements q, r and s are clearly of order two, so 14 entries can be filled in without any calculation. A few candidates thought t and u had order two.

b. Generally well done. A few candidates were unaware of the definition of the order of an element.

c. Often well done. A few candidates stated extra, and therefore incorrect subgroups.

The permutation p_1 of the set $\{1, 2, 3, 4\}$ is defined by

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

(a) (i) State the inverse of p_1.

(ii) Find the order of p_1.

(b) Another permutation p_2 is defined by

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

(i) Determine whether or not the composition of p_1 and p_2 is commutative.

(ii) Find the permutation p_3 which satisfies

$$p_1 p_3 p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Markscheme

(a) (i) the inverse is

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ ***A1***

(ii) **EITHER**

$1 \to 2 \to 4 \to 3 \to 1$ (is a cycle of length 4) ***R3***

so p_1 is of order 4 ***A1*** ***N2***

OR

consider

$p_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$ ***M1A1***

it is now clear that

$p_1^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ ***A1***

so p_1 is of order 4 ***A1*** ***N2***

[5 marks]

(b) (i) consider

$$p_1p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \textbf{\textit{M1A1}}$$

$$p_2p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \textbf{\textit{A1}}$$

composition is not commutative ***A1***

Note: In this part do not penalize candidates who incorrectly reverse the order both times.

(ii) **EITHER**

pre and postmultiply by p_1^{-1}, $p_2^{-1}$to give

$p_3 = p_1^{-1}p_2^{-1}$ ***(M1)(A1)***

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad \textbf{\textit{A1}}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \textbf{\textit{A1}}$$

OR

starting from

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \textbf{\textit{M1}}$$

successively deducing each missing number, to get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \textbf{\textit{A3}}$$

[8 marks]

Total [13 marks]

Examiners report

Many candidates scored well on this question although some gave the impression of not having studied this topic. The most common error in (b) was to believe incorrectly that p_1p_2 means p_1 followed by p_2. This was condoned in (i) but penalised in (ii). The Guide makes it quite clear that this is the notation to be used.

Let R be a relation on the set $\mathbb{Z}$ such that $aRb \Leftrightarrow ab \geqslant 0$, for $a, b \in \mathbb{Z}$.

(a) Determine whether R is

(i) reflexive;

(ii) symmetric;

(iii) transitive.

(b) Write down with a reason whether or not R is an equivalence relation.

Markscheme

(a) (i) $a^2 \geqslant 0$ for all $a \in \mathbb{Z}$, hence R is reflexive ***R1***

(ii) $aRb \Rightarrow ab \geqslant 0$ ***M1***

$\Rightarrow ba \geqslant 0$ ***R1***

$\Rightarrow bRa$, hence R is symmetric ***A1***

(iii) aRb and $bRc \Rightarrow ab \geqslant 0$ and $bc \geqslant 0,$ is aRc? ***M1***

no, for example, $-3R0$ and $0R5$, but $-3R5$ is not true ***A1***

aRc is not generally true, hence R is not transitive ***A1***

[7 marks]

(b) R does not satisfy all three properties, hence R is not an equivalence relation ***R1***

[1 mark]

Total [8 marks]

Examiners report

Although the properties of an equivalence relation were well known, few candidates provided a counter-example to show that the relation is not transitive. Some candidates interchanged the definitions of the reflexive and symmetric properties.

The relation R is defined for a , $b \in \mathbb{Z}^+$ such that aRb if and only if $a^2 - b^2$ is divisible by 5.

a. Show that R is an equivalence relation. [6]

b. Identify the three equivalence classes. [4]

Markscheme

a. reflexive: aRa because $a^2 - a^2 = 0$ (which is divisible by 5) ***A1***

symmetric: let aRb so that $a^2 - b^2 = 5M$ ***M1***
it follows that $a^2 - b^2 = -5M$ which is divisible by 5 so bRa ***A1***
transitive: let aRb and bRc so that $a^2 - b^2 = 5M$ and $b^2 - c^2 = 5N$ ***M1***
$a^2 - b^2 + b^2 - c^2 = 5M + 5N$ ***A1***
$a^2 - c^2 = 5M + 5N$ which is divisible by 5 so aRc ***A1***
$\Rightarrow R$ is an equivalence relation ***AG***
[6 marks]

b. the equivalence classes are

$\{1, 4, 6, 9, \ldots\}$ ***A2***

$\{2, 3, 7, 8, \ldots\}$ ***A1***

{5, 10, …}   ***A1***

Note: Do not award any marks for classes containing fewer elements than shown above.

[4 marks]

Examiners report

a. Many candidates solved (a) correctly but solutions to (b) were generally poor. Most candidates seemed to have a weak understanding of the concept of equivalence classes and were unaware of any systematic method for finding the equivalence classes. If all else fails, a trial and error approach can be used. Here, starting with 1, it is easily seen that 4, 6,… belong to the same class and the pattern can be established.

b. Many candidates solved (a) correctly but solutions to (b) were generally poor. Most candidates seemed to have a weak understanding of the concept of equivalence classes and were unaware of any systematic method for finding the equivalence classes. If all else fails, a trial and error approach can be used. Here, starting with 1, it is easily seen that 4, 6, … belong to the same class and the pattern can be established.

Let G be a group of order 12 with identity element e.

Let $a \in G$ such that $a^6 \neq e$ and $a^4 \neq e$.

a. (i) Prove that G is cyclic and state two of its generators. [9]

(ii) Let H be the subgroup generated by a^4. Construct a Cayley table for H.

b. State, with a reason, whether or not it is necessary that a group is cyclic given that all its proper subgroups are cyclic. [2]

Markscheme

a. (i) the order of a is a divisor of the order of G ***(M1)***

since the order of G is 12, the order of a must be 1, 2, 3, 4, 6 or 12   ***A1***
the order cannot be 1, 2, 3 or 6, since $a^6 \neq e$ ***R1***
the order cannot be 4, since $a^4 \neq e$ ***R1***
so the order of a must be 12
therefore, a is a generator of G, which must therefore be cyclic   ***R1***
another generator is *eg* $a^{-1},\ a^5,\ \ldots$ ***A1***
[6 marks]

(ii) $H = \{e,\ a^4,\ a^8\}$ ***(A1)***

	e	a^4	a^8
e	e	a^4	a^8
a^4	a^4	a^8	e
a^8	a^8	e	a^4

M1A1

[3 marks]

b. no ***A1***

eg the group of symmetries of a triangle S_3 is not cyclic but all its (proper) subgroups are cyclic
eg the Klein four-group is not cyclic but all its (proper) subgroups are cyclic ***R1***
[2 marks]

Examiners report

a. In part (a), many candidates could not provide a logical sequence of steps to show that G is cyclic. In particular, although they correctly quoted Lagrange's theorem, they did not always consider all the orders of a, i.e., all the factors of 12, omitting in particular 1 as a factor. Some candidates did not state the second generator, in particular a^{-1}. Very few candidates were successful in finding the required subgroup, although they were obviously familiar with setting up a Cayley table.

b.

(a) Show that $\{1, -1, \text{i}, -\text{i}\}$ forms a group of complex numbers G under multiplication.

(b) Consider $S = \{e,\ a,\ b,\ a * b\}$ under an associative operation $*$ where e is the identity element. If $a * a = b * b = e$ and $a * b = b * a$, show that

(i) $a * b * a = b$,

(ii) $a * b * a * b = e$.

(c) (i) Write down the Cayley table for $H = \{S\ ,*\}$.

(ii) Show that H is a group.

(iii) Show that H is an Abelian group.

(d) For the above groups, G and H , show that one is cyclic and write down why the other is not. Write down all the generators of the cyclic group.

(e) Give a reason why G and H are not isomorphic.

Markscheme

(a)

	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

see the Cayley table, (since there are no new elements) the set is closed ***A1***
1 is the identity element ***A1***
1 and –1 are self inverses and i and -i form an inverse pair, hence every element has an inverse ***A1***
multiplication is associative ***A1***
hence $\{1, –1, \text{i}, –\text{i}\}$ form a group G under the operation of multiplication ***AG***
[4 marks]

(b) (i) $aba = aab$

$= eb$ ***A1***

$= b$ ***AG***

(ii) $abab = aabb$

$= ee$ ***A1***

$= e$ ***AG***

[2 marks]

(c) (i)

$*$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

A2

Note: Award ***A1*** for 1 or 2 errors, ***A0*** for more than 2.

(ii) see the Cayley table, (since there are no new elements) the set is closed ***A1***

H has an identity element e ***A1***

all elements are self inverses, hence every element has an inverse ***A1***

the operation is associative as stated in the question

hence $\{e, a, b, ab\}$ forms a group G under the operation $*$ ***AG***

(iii) since there is symmetry across the leading diagonal of the group table, the group is Abelian ***A1***

[6 marks]

(d) consider the element i from the group G ***(M1)***

$\mathrm{i}^2 = -1$

$\mathrm{i}^3 = -\mathrm{i}$

$\mathrm{i}^4 = 1$

thus i is a generator for G and hence G is a cyclic group ***A1***

–i is the other generator for G ***A1***

for the group H there is no generator as all the elements are self inverses ***R1***

[4 marks]

(e) since one group is cyclic and the other group is not, they are not isomorphic ***R1***

[1 mark]

Total [17 marks]

Examiners report

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. A number of candidates did not understand the term “Abelian”. Many candidates understood the conditions for a group to be cyclic. Many candidates did not realise that the answer to part (e) was actually found in part (d), hence the reason for this part only being worth 1 mark. Overall, a number of fully correct solutions to this question were seen.

The relations R and S are defined on quadratic polynomials P of the form

$$P(z) = z^2 + az + b \text{, where } a \text{ , } b \in \mathbb{R} \text{ , } z \in \mathbb{C} \text{ .}$$

(a) The relation R is defined by P_1RP_2 if and only if the sum of the two zeros of P_1 is equal to the sum of the two zeros of P_2 .

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence class containing $z^2 - 4z + 5$.

(b) The relation S is defined by P_1SP_2 if and only if P_1 and P_2 have at least one zero in common. Determine whether or not S is transitive.

Markscheme

(a) (i) R is reflexive, *i.e.* PRP because the sum of the zeroes of P is equal to the sum of the zeros of P ***R1***

R is symmetric, *i.e.* $P_1RP_2 \Rightarrow P_2RP_1$ because the sums of the zeros of P_1 and P_2 are equal implies that the sums of the zeros of P_2 and P_1 are equal ***R1***

suppose that P_1RP_2 and P_2RP_3 ***M1***

it follows that P_1RP_3 so R is transitive, because the sum of the zeros of P_1 is equal to the sum of the zeros of P_2 which in turn is equal to the sum of the zeros of P_3 , which implies that the sum of the zeros of P_1 is equal to the sum of the zeros of P_3 ***R1***

the three requirements for an equivalence relation are therefore satisfied ***AG***

(ii) the zeros of $z^2 - 4z + 5$ are $2 \pm \mathrm{i}$, for which the sum is 4 ***M1A1***

$\mathrm{z}^2 + az + b$ has zeros of $\frac{-a\pm\sqrt{a^2-4b}}{2}$, so the sum is $-a$ ***(M1)***

Note: Accept use of the result (although not in the syllabus) that the sum of roots is minus the coefficient of z.

hence $-a = 4$ and so $a = -4$ ***A1***

the equivalence class is $z^2 - 4z + k$, $(k \in \mathbb{R})$ ***A1***

[9 marks]

(b) for example, $(z-1)(z-2)S(z-1)(z-3)$ and

$(z-1)(z-3)S(z-3)(z-4)$ but $(z-1)(z-2)S(z-3)(z-4)$ is not true ***M1A1***

so S is not transitive ***A1***

[3 marks]

Total [12 marks]

Examiners report

Most candidates were able to show, in (a), that R is an equivalence relation although few were able to identify the required equivalence class. In (b), the explanation that S is not transitive was often unconvincing.

The relation R is defined on ordered pairs by

$$(a,\ b)R(c,\ d) \text{ if and only if } ad = bc \text{ where } a,\ b,\ c,\ d \in \mathbb{R}^+.$$

(a) Show that R is an equivalence relation.

(b) Describe, geometrically, the equivalence classes.

Markscheme

(a) Reflexive: $(a,\ b)R(a,\ b)$ because $ab = ba$ ***R1***

Symmetric: $(a,\ b)R(c,\ d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,\ d)R(a,\ b)$ ***M1A1***
Transitive: $(a,\ b)R(c,\ d) \Rightarrow ad = bc$ ***M1***
$(c,\ d)R(e,\ f) \Rightarrow cf = de$
Therefore
$\frac{ad}{de} = \frac{bc}{cf}$ so $af = be$ ***A1***
It follows that $(a,\ b)R(e,\ f)$ ***R1***
[6 marks]

(b) $(a,\ b)R(c,\ d) \Rightarrow \frac{a}{b} = \frac{c}{d}$ ***(M1)***
Equivalence classes are therefore points lying, in the first quadrant, on straight lines through the origin. ***A2***
Notes: Accept a correct sketch.
Award ***A1*** if "in the first quadrant" is omitted.
Do not penalise candidates who fail to exclude the origin.

[3 marks]
Total [9 marks]

Examiners report

Part (a) was well answered by many candidates although some misunderstandings of the terminology were seen. Some candidates appeared to believe, incorrectly, that reflexivity was something to do with $(a,\ a)R(a,\ a)$ and some candidates confuse the terms 'reflexive' and 'symmetric'. Many candidates were unable to describe the equivalence classes geometrically.

Consider the set $S = \{1, 3, 5, 7, 9, 11, 13\}$ under the binary operation multiplication modulo 14 denoted by $\times_{14}$.

a. Copy and complete the following Cayley table for this binary operation. [4]

$\times_{14}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**
1	1	3	5	7	9	11	13
3	3				13	5	11
5	5				3	13	9
7	7						
9	9	13	3				
11	11	5	13				
13	13	11	9				

b. Give one reason why $\{S,\ \times_{14}\}$ is not a group. [1]

c. Show that a new set G can be formed by removing one of the elements of S such that $\{G,\ \times_{14}\}$ is a group. [5]

d. Determine the order of each element of $\{G,\ \times_{14}\}$. [4]

e. Find the proper subgroups of $\{G,\ \times_{14}\}$. [2]

Markscheme

a.

$\times_{14}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**
1	1	3	5	7	9	11	13
3	3	9	1	7	13	5	11
5	5	1	11	7	3	13	9
7	7	7	7	7	7	7	7
9	9	13	3	7	11	1	5
11	11	5	13	7	1	9	3
13	13	11	9	7	5	3	1

A4

Note: Award ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors, ***A0*** for four or more errors.

[4 marks]

b. any valid reason, for example ***R1***

not a Latin square

7 has no inverse

[1 mark]

c. delete 7 (so that $G = \{1, 3, 5, 9, 11, 13\}$) ***A1***

closure – evident from the table ***A1***

associative because multiplication is associative ***A1***

the identity is 1 ***A1***

13 is self-inverse, 3 and 5 form an inverse

pair and 9 and 11 form an inverse pair ***A1***

the four conditions are satisfied so that $\{G,\ \times_{14}\}$ is a group ***AG***

[5 marks]

d.

Element	Order
1	1
3	6
5	6
9	3
11	3
13	2

A4

Note: Award ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors, ***A0*** for four or more errors.

[4 marks]

e. $\{1\}$

$\{1, 13\}$ $\{1, 9, 11\}$ ***A1A1***

[2 marks]

Examiners report

a. There were no problems with parts (a), (b) and (d).

b. There were no problems with parts (a), (b) and (d).

c. There were no problems with parts (a), (b) and (d) but in part (c) candidates often failed to state that the set was associative under the operation because multiplication is associative. Likewise they often failed to list the inverses of each element simply stating that the identity was present in each row and column of the Cayley table.

d. The majority of candidates did not answer part (d) correctly and often simply listed all subsets of order 2 and 3 as subgroups.

e. [N/A]

The binary operator multiplication modulo 14, denoted by $*$, is defined on the set $S = \{2, 4, 6, 8, 10, 12\}$.

a. Copy and complete the following operation table. [4]

$*$	2	4	6	8	10	12
2						
4	8	2	10	4	12	6
6						
8						
10	6	12	4	10	2	8
12						

b. (i) Show that $\{S, *\}$ is a group. [11]

(ii) Find the order of each element of $\{S, *\}$.

(iii) Hence show that $\{S, *\}$ is cyclic and find all the generators.

c. The set T is defined by $\{x * x : x \in S\}$. Show that $\{T, *\}$ is a subgroup of $\{S, *\}$. [3]

Markscheme

a.

$*$	2	4	6	8	10	12
2	4	8	12	2	6	10
4	8	2	10	4	12	6
6	12	10	8	6	4	2
8	2	4	6	8	10	12
10	6	12	4	10	2	8
12	10	6	2	12	8	4

A4

Note: Award *A4* for all correct, *A3* for one error, *A2* for two errors, *A1* for three errors and *A0* for four or more errors.

[4 marks]

b. (i) closure: there are no new elements in the table ***A1***

identity: 8 is the identity element ***A1***

inverse: every element has an inverse because there is an 8 in every row and column ***A1***

associativity: (modulo) multiplication is associative ***A1***

therefore $\{S, *\}$ is a group ***AG***

(ii) the orders of the elements are as follows

element	order
2	3
4	3
6	2
8	1
10	6
12	6

A4

Note: Award ***A4*** for all correct, ***A3*** for one error, ***A2*** for two errors, ***A1*** for three errors and ***A0*** for four or more errors.

(iii) **EITHER**

the group is cyclic because there are elements of order 6 ***R1***

OR

the group is cyclic because there are generators ***R1***

THEN

10 and 12 are the generators ***A1A1***

[11 marks]

c. looking at the Cayley table, we see that

$T = \{2, 4, 8\}$ ***A1***

this is a subgroup because it contains the identity element 8, no new elements are formed and 2 and 4 form an inverse pair ***R2***

Note: Award ***R1*** for any two conditions

[3 marks]

Examiners report

a. Parts (a) and (b) were well done in general. Some candidates, however, when considering closure and associativity simply wrote ‘closed’ and ‘associativity’ without justification. Here, candidates were expected to make reference to their Cayley table to justify closure and to state that multiplication is associative to justify associativity. In (c), some candidates tried to show the required result without actually identifying the elements of T. This approach was invariably unsuccessful.

b. Parts (a) and (b) were well done in general. Some candidates, however, when considering closure and associativity simply wrote ‘closed’ and ‘associativity’ without justification. Here, candidates were expected to make reference to their Cayley table to justify closure and to state that multiplication is associative to justify associativity. In (c), some candidates tried to show the required result without actually identifying the elements of T. This approach was invariably unsuccessful.

c. Parts (a) and (b) were well done in general. Some candidates, however, when considering closure and associativity simply wrote ‘closed’ and ‘associativity’ without justification. Here, candidates were expected to make reference to their Cayley table to justify closure and to state that multiplication is associative to justify associativity. In (c), some candidates tried to show the required result without actually identifying the elements of T. This approach was invariably unsuccessful.

The group $\{G,\ *\}$ is defined on the set G with binary operation $*$. H is a subset of G defined by $H=\{x:\ x\in G,\ a*x*a^{-1}=x \text{ for all } a\in G\}$. Prove that $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$.

Markscheme

associativity: This follows from associativity in $\{G,\ *\}$ ***R1***

the identity $e\in H$ since $a*e*a^{-1}=a*a^{-1}=e$ (for all $a\in G$) ***R1***

Note: Condone the use of the commutativity of *e* if that is involved in an alternative simplification of the LHS.

closure: Let $x,\ y\in H$ so that $a*x*a^{-1}=x$ and $a*y*a^{-1}=y$ for all $a\in G$ ***(M1)***

multiplying, $x*y=a*x*a^{-1}*a*y*a^{-1}$ (for all $a\in G$) ***A1***

$=a*x*y*a^{-1}$ ***A1***

therefore $x*y\in H$ (proving closure) ***R1***

inverse: Let $x\in H$ so that $a*x*a^{-1}=x$ (for all $a\in G$)

$x^{-1}=(a*x*a^{-1})^{-1}$ ***M1***

$=a*x^{-1}*a^{-1}$ ***A1***

therefore $x^{-1}\in H$ ***R1***

hence $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$ ***AG***

Note: Accuracy marks cannot be awarded if commutativity is assumed for general elements of G.

[9 marks]

Examiners report

This is an abstract question, clearly defined on a subset. Far too many candidates almost immediately deduced, erroneously, that the full group was Abelian. Almost no marks were then available.

The following Cayley table for the binary operation multiplication modulo 9, denoted by $*$, is defined on the set $S=\{1,\ 2,\ 4,\ 5,\ 7,\ 8\}$.

*	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8				
5	5	1				
7	7	5				
8	8	7				

a. Copy and complete the table. [3]

b. Show that $\{S,\ *\}$ is an Abelian group. [5]

c. Determine the orders of all the elements of $\{S,\ *\}$. [3]

d. (i) Find the two proper subgroups of $\{S,\ *\}$. [4]

(ii) Find the coset of each of these subgroups with respect to the element 5.

e. Solve the equation $2 * x * 4 * x * 4 = 2$. [4]

Markscheme

a.

*	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

A3

Note: Award ***A3*** for correct table, ***A2*** for one or two errors, ***A1*** for three or four errors and ***A0*** otherwise.

[3 marks]

b. the table contains only elements of S, showing closure ***R1***

the identity is 1 ***A1***

every element has an inverse since 1 appears in every row and column, or a complete list of elements and their correct inverses ***A1***

multiplication of numbers is associative ***A1***

the four axioms are satisfied therefore $\{S,\ *\}$ is a group

the group is Abelian because the table is symmetric (about the leading diagonal) ***A1***

[5 marks]

c.

Element	Order
1	1
2	6
4	3
5	6
7	3
8	2

A3

Note: Award ***A3*** for all correct values, ***A2*** for 5 correct, ***A1*** for 4 correct and ***A0*** otherwise.

[3 marks]

d. (i) the subgroups are $\{1,\ 8\}$; $\{1,\ 4,\ 7\}$ ***A1A1***

(ii) the cosets are $\{4,\ 5\}$; $\{2,\ 5,\ 8\}$ ***A1A1***

[4 marks]

e. **METHOD 1**

use of algebraic manipulations ***M1***

and at least one result from the table, used correctly ***A1***

$x = 2$ ***A1***

$x = 7$ ***A1***

METHOD 2

testing at least one value in the equation ***M1***

obtain $x = 2$ ***A1***

obtain $x = 7$ ***A1***

explicit rejection of all other values ***A1***

[4 marks]

Examiners report

a. The majority of candidates were able to complete the Cayley table correctly.

b. Generally well done. However, it is not good enough for a candidate to say something along the lines of 'the operation is closed or that inverses exist by looking at the Cayley table'. A few candidates thought they only had to prove commutativity.

c. Often well done. A few candidates stated extra, and therefore incorrect subgroups.

d. [N/A]

e. The majority found only one solution, usually the obvious $x = 2$, but sometimes only the less obvious $x = 7$.

The binary operation multiplication modulo 10, denoted by $\times_{10}$, is defined on the set T = {2 , 4 , 6 , 8} and represented in the following Cayley table.

$\times_{10}$	**2**	**4**	**6**	**8**
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

a. Show that $\{T, \times_{10}\}$ is a group. (You may assume associativity.) [4]

b. By making reference to the Cayley table, explain why T is Abelian. [1]

c.i. Find the order of each element of $\{T, \times_{10}\}$. [3]

c.ii. Hence show that $\{T, \times_{10}\}$ is cyclic and write down all its generators. [3]

d. The binary operation multiplication modulo 10, denoted by $\times_{10}$, is defined on the set $V = \{1, 3, 5, 7, 9\}$. [2]

Show that $\{V, \times_{10}\}$ is not a group.

Markscheme

a. closure: there are no new elements in the table ***A1***

identity: 6 is the identity element ***A1***

inverse: every element has an inverse because there is a 6 in every row and column ($2^{-1} = 8$, $4^{-1} = 4$, $6^{-1} = 6$, $8^{-1} = 2$) ***A1***

we are given that (modulo) multiplication is associative ***R1***

so $\{T, \times_{10}\}$ is a group ***AG***

[4 marks]

b. the Cayley table is symmetric (about the main diagonal) ***R1***

so T is Abelian ***AG***

[1 mark]

c.i. considering powers of elements ***(M1)***

elements	order
2	4
4	2
6	1
8	4

A2

Note: Award ***A2*** for all correct and ***A1*** for one error.

[3 marks]

c.ii. **EITHER**

$\{T, \times_{10}\}$ is cyclic because there is an element of order 4 ***R1***

Note: Accept "there are elements of order 4".

OR

$\{T, \times_{10}\}$ is cyclic because there is generator ***R1***

Note: Accept "because there are generators".

THEN

2 and 8 are generators ***A1A1***

[3 marks]

d. **EITHER**

considering singular elements ***(M1)***

5 has no inverse ($5 \times_{10} a = 1$, $a \in V$ has no solution) ***R1***

OR

considering Cayley table for $\{V, \times_{10}\}$

$\times_{10}$	1	3	5	7	9
1	1	3	5	7	9
3	3	9	5	1	7
5	5	5	5	5	5
7	7	1	5	9	3
9	9	7	5	3	1

M1

the Cayley table is not a Latin square (or equivalent) ***R1***

OR

considering cancellation law

eg, $5 \times_{10} 9 = 5 \times_{10} 1 = 5$ ***M1***

if $\{V, \times_{10}\}$ is a group the cancellation law gives 9 = 1 ***R1***

OR

considering order of subgroups

eg, {1, 9} is a subgroup ***M1***

it is not possible to have a subgroup of order 2 for a group of order 5 (Lagrange's theorem) ***R1***

THEN

so $\{V, \times_{10}\}$ is not a group ***AG***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c.i. [N/A]
c.ii. [N/A]
d. [N/A]

Consider the sets $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7, 11\}$ and $C = \{1, 3, 7, 15, 31\}$.

a.i. Find $(A \cup B) \cap (A \cup C)$. [3]

a.ii. Verify that $A \setminus C \neq C \setminus A$. [2]

b. Let S be a set containing n elements where $n \in \mathbb{N}$. [3]

Show that S has 2^n subsets.

Markscheme

a.i. **EITHER**

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 5, 7, 9, 11\} \cap \{1, 3, 5, 7, 9, 15, 31\}$ ***M1A1***

OR

$A \cup (B \cap C) = \{1, 3, 5, 7, 9, 11\} \cup \{3, 7\}$ ***M1A1***

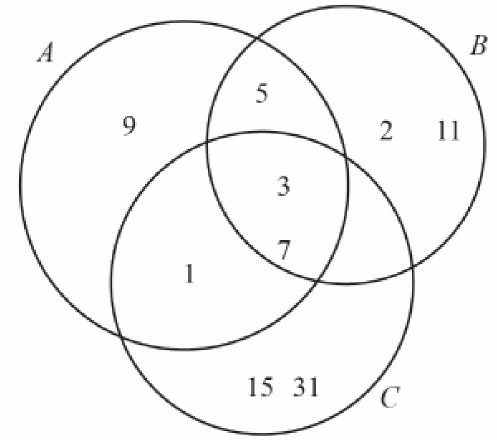
OR

$B \cap C$ is contained within *A* ***(M1)A1***

THEN

= {1, 3, 5, 7, 9} (= *A*) ***A1***

Note: Accept a Venn diagram representation.



[3 marks]

a.ii. *A* \ *C* = {5, 9} ***A1***

C \ *A* = {15, 31} ***A1***

so *A* \ *C* ≠ *C* \ *A* ***AG***

Note: Accept a Venn diagram representation.

[2 marks]

b. **METHOD 1**

if $S = \emptyset$ then $n = 0$ and the number of subsets of *S* is given by $2^0 = 1$ ***A1***

if $n > 0$

for every subset of *S*, there are 2 possibilities for each element $x \in S$ either x will be in the subset or it will not ***R1***

so for all n elements there are $(2 \times 2 \times \ldots \times 2)\, 2^n$ different choices in forming a subset of *S* ***R1***

so *S* has 2^n subsets ***AG***

Note: If candidates attempt induction, award ***A1*** for case $n = 0$, ***R1*** for setting up the induction method (assume $P(k)$ and consider $P(k+1)$ and ***R1*** for showing how the $P(k)$ true implies $P(k+1)$ true).

METHOD 2

$\sum\limits_{k=0}^{n} \binom{n}{k}$ is the number of subsets of *S* (of all possible sizes from 0 to n) ***R1***

$(1+1)^n = \sum\limits_{k=0}^{n} \binom{n}{k} \left(1^k\right) \left(1^{n-k}\right)$ ***M1***

$2^n = \sum\limits_{k=0}^{n} \binom{n}{k}$ (= number of subsets of *S*) ***A1***

so *S* has 2^n subsets ***AG***

[3 marks]

Examiners report

a.i. [N/A]

a.ii. [N/A]
b. [N/A]

---

a. Consider the following Cayley table for the set $G = \{1, 3, 5, 7, 9, 11, 13, 15\}$ under the operation $\times_{16}$, where $\times_{16}$ denotes multiplication modulo 16. [7]

$\times_{16}$	1	3	5	7	9	11	13	15
1	1	3	5	7	9	11	13	15
3	3	a	15	5	11	b	7	c
5	5	15	9	3	13	7	1	11
7	7	d	3	1	e	13	f	9
9	9	11	13	g	1	3	5	7
11	11	h	7	13	3	9	i	5
13	13	7	1	11	5	j	9	3
15	15	13	11	9	7	5	3	1

(i) Find the values of $a, b, c, d, e, f, g, h, i$ and j.

(ii) Given that $\times_{16}$ is associative, show that the set G, together with the operation $\times_{16}$, forms a group.

b. The Cayley table for the set $H = \{e,\ a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3,\ b_4\}$ under the operation $*$, is shown below. [8]

$*$	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
e	e	a_1	a_2	a_3	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	e	b_4	b_3	b_1	b_2
a_2	a_2	a_3	e	a_1	b_2	b_1	b_4	b_3
a_3	a_3	e	a_1	a_2	b_3	b_4	b_2	b_1
b_1	b_1	b_3	b_2	b_4	e	a_2	a_1	a_3
b_2	b_2	b_4	b_1	b_3	a_2	e	a_3	a_1
b_3	b_3	b_2	b_4	b_1	a_3	a_1	e	a_2
b_4	b_4	b_1	b_3	b_2	a_1	a_3	a_2	e

(i) Given that $*$ is associative, show that H together with the operation $*$ forms a group.

(ii) Find two subgroups of order 4.

c. Show that $\{G,\ \times_{16}\}$ and $\{H,\ *\}$ are not isomorphic. [2]

d. Show that $\{H,\ *\}$ is not cyclic. [3]

Markscheme

a. (i) $a = 9,\ b = 1,\ c = 13,\ d = 5,\ e = 15,\ f = 11,\ g = 15,\ h = 1,\ i = 15,\ j = 15$ ***A3***

Note: Award ***A2*** for one or two errors,

A1 for three or four errors,

A0 for five or more errors.

(ii) since the Cayley table only contains elements of the set G, then it is closed ***A1***

there is an identity element which is 1 ***A1***

$\{3, 11\}$ and $\{5, 13\}$ are inverse pairs and all other elements are self inverse ***A1***

hence every element has an inverse ***R1***

Note: Award ***A0R0*** if no justification given for every element having an inverse.

since the set is closed, has an identity element, every element has an inverse and it is associative, it is a group ***AG***

[7 marks]

b. (i) since the Cayley table only contains elements of the set H, then it is closed ***A1***

there is an identity element which is e ***A1***

$\{a_1, a_3\}$ form an inverse pair and all other elements are self inverse ***A1***

hence every element has an inverse ***R1***

Note: Award ***A0R0*** if no justification given for every element having an inverse.

since the set is closed, has an identity element, every element has an inverse and it is associative, it is a group ***AG***

(ii) any 2 of $\{e, a_1, a_2, a_3\}$, $\{e, a_2, b_1, b_2\}$, $\{e, a_2, b_3, b_4\}$ ***A2A2***

[8 marks]

c. the groups are not isomorphic because $\{H, *\}$ has one inverse pair whereas $\{G, \times_{16}\}$ has two inverse pairs ***A2***

Note: Accept any other valid reason:

e.g. the fact that $\{G, \times_{16}\}$ is commutative and $\{H, *\}$ is not.

[2 marks]

d. **EITHER**

a group is not cyclic if it has no generators ***R1***

for the group to have a generator there must be an element in the group of order eight ***A1***

element	order
e	1
a_1	4
a_2	2
a_3	4
b_1	2
b_2	2
b_3	2
b_4	2

since there is no element of order eight in the group, it is not cyclic ***A1***

OR

a group is not cyclic if it has no generators ***R1***

only possibilities are a_1, a_3 since all other elements are self inverse ***A1***

this is not possible since it is not possible to generate any of the "*b*" elements from the "*a*" elements – the elements $a_1,\ a_2,\ a_3,\ a_4$ form a closed set ***A1***

[3 marks]

Examiners report

a. Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

b. Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

c. Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

d. Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

---

a. The function $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(n) = |n| - 1$ for $n \in \mathbb{Z}$. Show that g is neither surjective nor injective. [2]

b. The set S is finite. If the function $f : S \to S$ is injective, show that *f* is surjective. [2]

c. Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. [3]

Markscheme

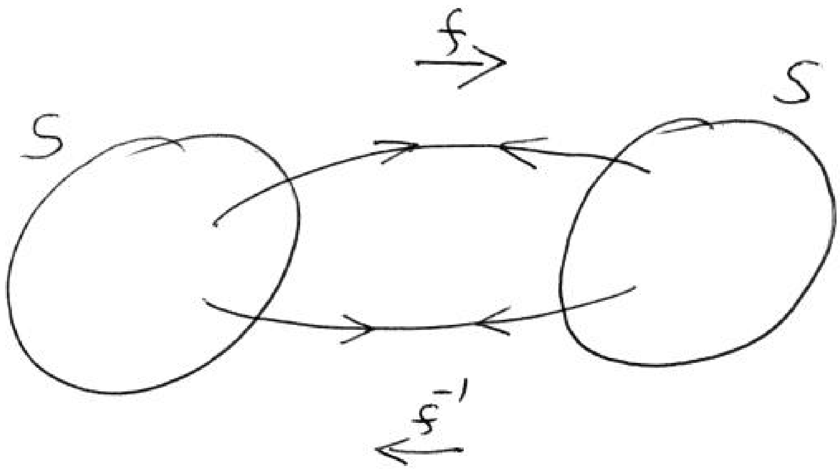
a. non-S: for example –2 does not belong to the range of g ***R1***

non-I: for example $g(1) = g(-1) = 0$ ***R1***

Note: Graphical arguments have to recognize that we are dealing with sets of integers and not all real numbers

[2 marks]

b. as f is injective $n\left(f(S)\right) = n(S)$ ***A1***



R1

Note: Accept alternative explanations.

f is surjective ***AG***
[2 marks]

c. for example, $h(n) = n + 1$ ***A1***

Note: Only award the ***A1*** if the function works.

I: $n + 1 = m + 1 \Rightarrow n = m$ ***R1***
non-S: 1 has no pre-image as $0 \notin \mathbb{Z}^+$ ***R1***
[3 marks]

Examiners report

a. Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

b. Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

c. Nearly all candidates were aware of the conditions for an injection and a surjection in part (a). However, many missed the fact that the function in question was mapping from the set of integers to the set of integers. This led some to lose marks by applying graphical tests that were relevant for functions on the real numbers but not appropriate in this case. However, many candidates were able to give two integer counter

examples to prove that the function was neither injective or surjective. In part (b) candidates seemed to lack the communication skills to adequately demonstrate what they intuitively understood to be true. It was usually not stated that the number of elements in the sets of the image and pre – image was equal. Part (c) was well done by many candidates although a significant minority used functions that mapped the positive integers to non – integer values and thus not appropriate for the conditions required of the function.

Consider the functions $f : A \to B$ and $g : B \to C$.

a. Show that if both *f* and *g* are injective, then $g \circ f$ is also injective. [3]

b. Show that if both *f* and *g* are surjective, then $g \circ f$ is also surjective. [4]

c. Show, using a single counter example, that both of the converses to the results in part (a) and part (b) are false. [3]

Markscheme

a. let *s* and *t* be in *A* and $s \neq t$ ***M1***

since *f* is injective $f(s) \neq f(t)$ ***A1***

since *g* is injective $g \circ f(s) \neq g \circ f(t)$ ***A1***

hence $g \circ f$ is injective ***AG***

[3 marks]

b. let *z* be an element of *C*

we must find *x* in *A* such that $g \circ f(x) = z$ ***M1***

since *g* is surjective, there is an element *y* in *B* such that $g(y) = z$ ***A1***

since *f* is surjective, there is an element *x* in *A* such that $f(x) = y$ ***A1***

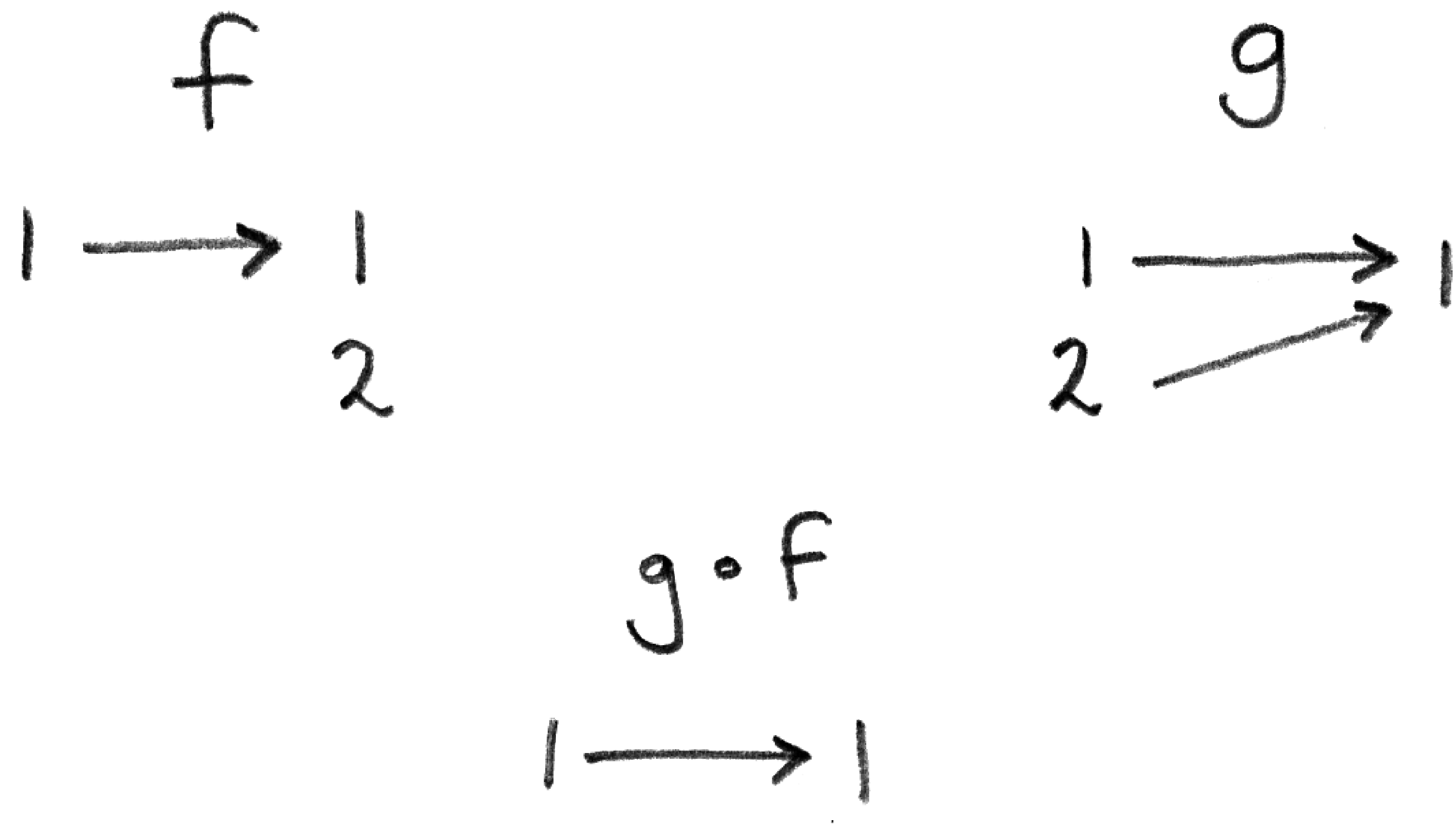
thus $g \circ f(x) = g(y) = z$ ***R1***

hence $g \circ f$ is surjective ***AG***

[4 marks]

c. converses: if $g \circ f$ is injective then *g* and *f* are injective

if $g \circ f$ is surjective then *g* and *f* are surjective ***(A1)***



A2

Note: There will be many alternative counter-examples.

[3 marks]

Examiners report

a. This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

b. This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

c. This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2x+1 & \text{for } x \leqslant 2 \\ x^2 - 2x + 5 & \text{for } x > 2. \end{cases}$$

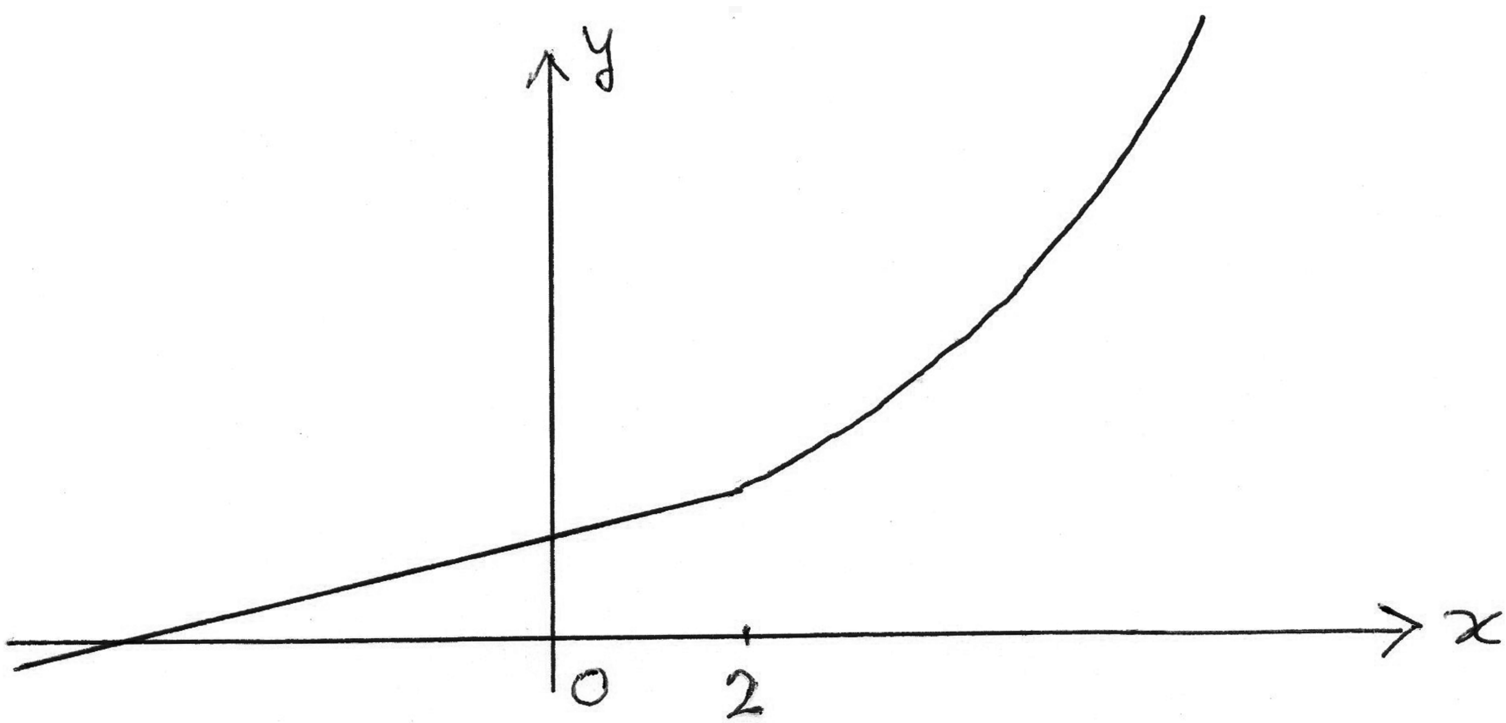
a. (i) Sketch the graph of *f*. [5]

(ii) By referring to your graph, show that *f* is a bijection.

b. Find $f^{-1}(x)$. [8]

Markscheme

a. (i)



A1A1

Note: Award ***A1*** for each part of the piecewise function. Award ***A1A0*** if the two parts of the graph are of the correct shape but *f* is not continuous at *x* = 2. Do not penalise a discontinuity in the derivative at *x* = 2.

(ii) demonstrating the need to show that *f* is both an injection and a surjection (seen anywhere) ***(R1)***

f is an injection by any valid reason *eg* horizontal line test, strictly increasing function ***R1***

the range of *f* is $\mathbb{R}$ so that *f* is a surjection ***R1***

f is therefore a bijection ***AG***

[5 marks]

b. considering the linear section, put

$y = 2x + 1$ or $x = 2y + 1$ ***(M1)***

$x = \frac{y-1}{2}$ or $y = \frac{x-1}{2}$ ***A1***

so $f^{-1}(x) = \frac{x-1}{2},\ x \leqslant 5$ ***A1***

EITHER

$y = (x-1)^2 + 4$ ***M1A1***

$(x-1)^2 = y - 4$

$x = 1 \pm \sqrt{y-4}$ ***A1***

$x = 1 + \sqrt{y-4}$

taking the + sign to give the right hand half of the parabola ***R1***

so $f^{-1}(x) = 1 + \sqrt{x-4},\ x > 5$ ***A1***

OR

considering the quadratic section, put

$y = x^2 - 2x + 5$

$x^2 - 2x + 5 - y = 0$ ***M1***

$x = \frac{2 \pm \sqrt{4-4(5-y)}}{2}\ (= 1 \pm \sqrt{y-4})$ ***M1A1***

taking the + sign to give the right hand half of the parabola ***R1***

so $f^{-1}(x) = \frac{2+\sqrt{4-4(5-x)}}{2},\ x > 5\ (f^{-1}(x) = 1 + \sqrt{x-4},\ x > 5)$ ***A1***

Note: Award ***A0*** for omission of $f^{-1}(x)$ or omission of the domain. Penalise the omission of the notation $f^{-1}(x)$ only once. The domain must be seen in both cases.

[8 marks]

Examiners report

a. For the most part the piecewise function was correctly graphed. Even though the majority of candidates knew that it is required to establish that the function is an injection and a surjection in order to prove it is a bijection, many just quoted the definition of injection or surjection and did not relate their reason to the graph.

b. The majority of candidates found the inverse of the first part of the piecewise function but some struggled with the algebra of the second part. In finding the inverse of the quadratic part of the function some candidates omitted the plus or minus sign in front of the square root. Others who had it often forgot to eliminate the negative sign and so did not gain the reasoning mark. Most did not state the correct domain for either part of the inverse function.

---

a. Determine, using Venn diagrams, whether the following statements are true. [6]

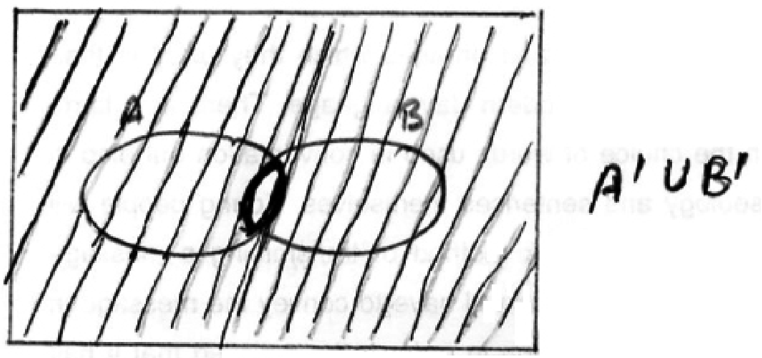
(i) $A' \cup B' = (A \cup B)'$

(ii) $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$

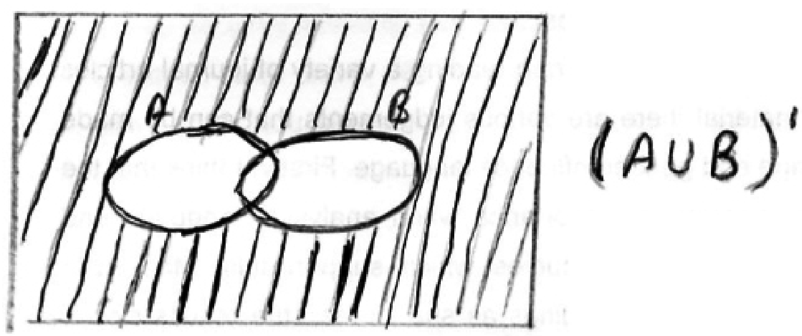
b. Prove, without using a Venn diagram, that $A \backslash B$ and $B \backslash A$ are disjoint sets. [4]

Markscheme

a. (a) (i)



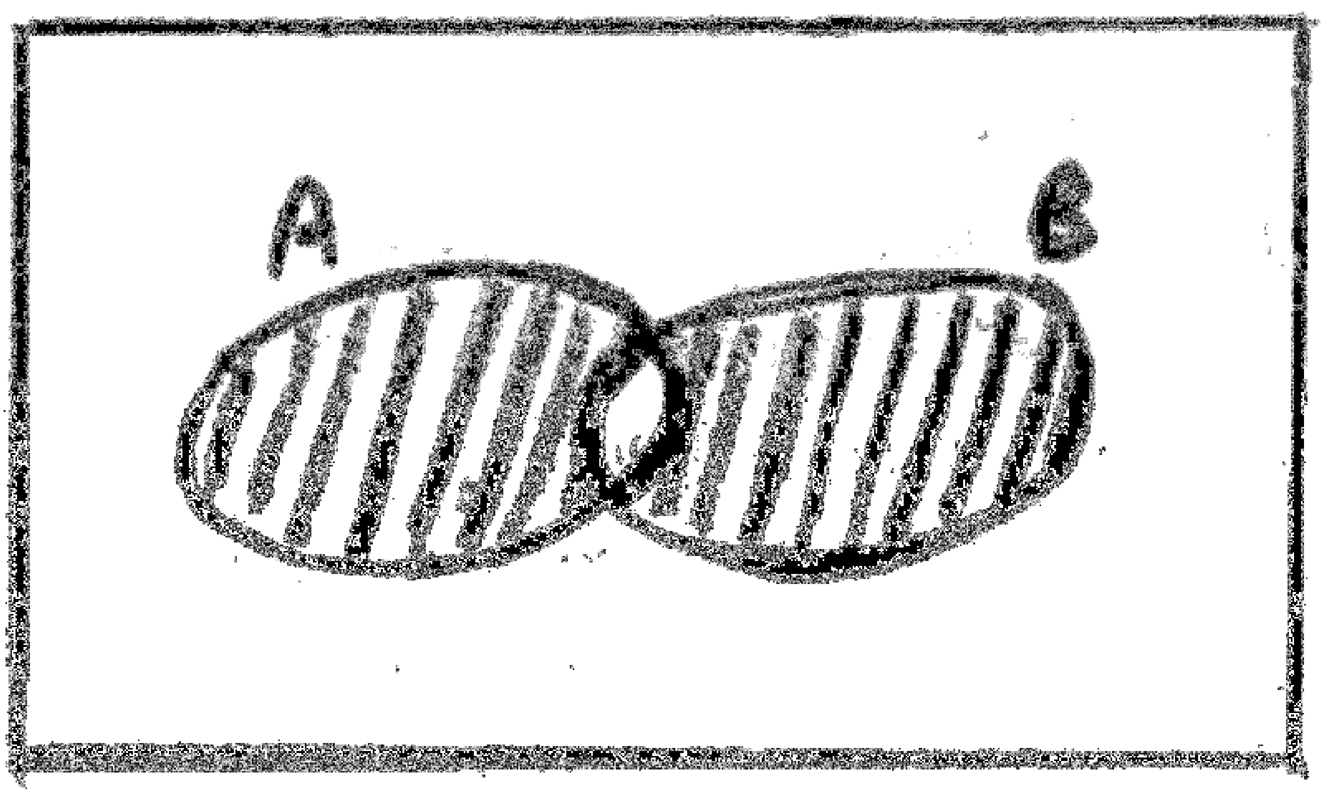
A1 ***A1***



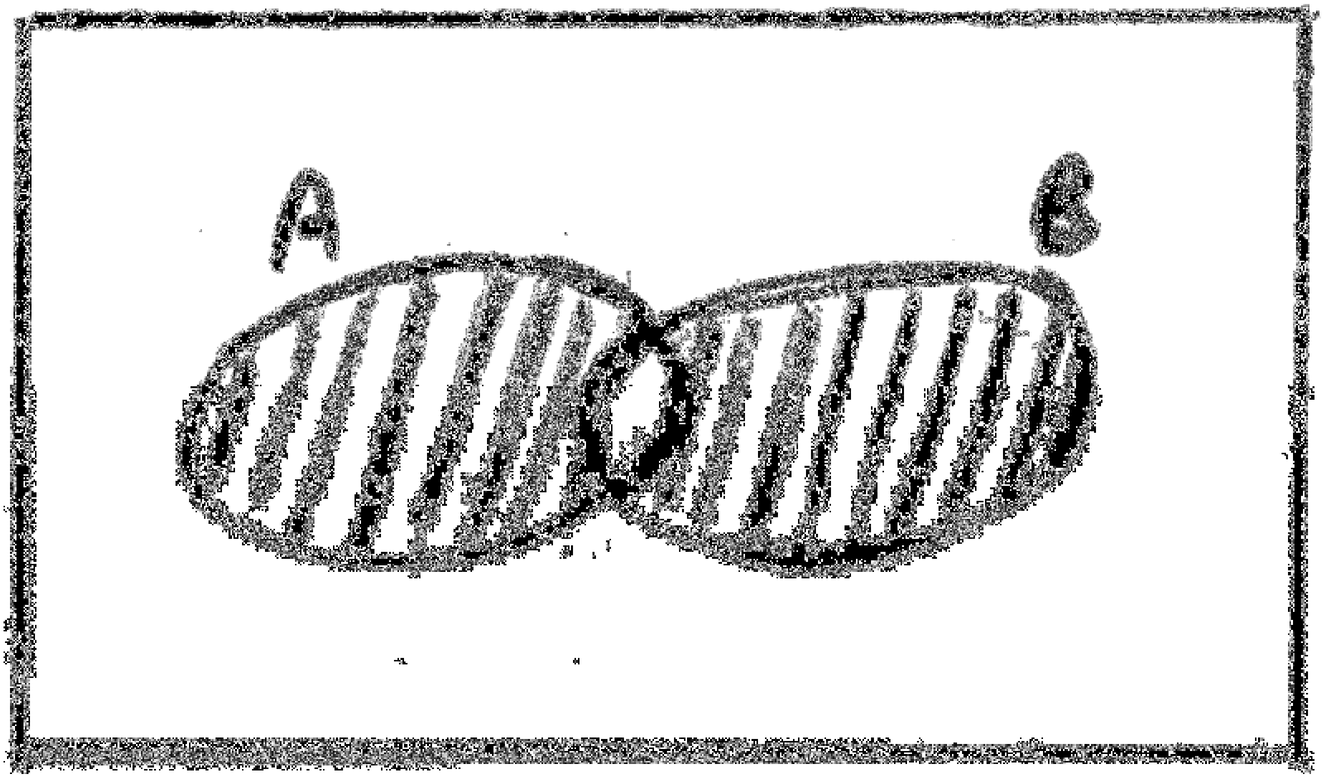
since the shaded regions are different, $A' \cup B' \neq (A \cup B)'$ ***R1***

$\Rightarrow$ not true

(ii)



A1



A1

since the shaded regions are the same $(A\backslash B) \cup (B\backslash A) = (A \cup B)\backslash(A \cap B)$ ***R1***

$\Rightarrow$ true

[6 marks]

b. $A\backslash B = A \cup B'$ and $B\backslash A = B \cap A'$ ***(A1)***

consider $A \cap B' \cap B \cap A'$ ***M1***

now $A \cap B' \cap B \cap A' = \emptyset$ ***A1***

since this is the empty set, they are disjoint ***R1***

Note: Accept alternative valid proofs.

[4 marks]

Examiners report

a. Part (a) was accessible to most candidates, but a number drew incorrect Venn diagrams. In some cases the clarity of the diagram made it difficult to follow what the candidate intended. Candidates found (b) harder, although the majority made a reasonable start to the proof. Once again a number of candidates were let down by poor explanation.

b. Part (a) was accessible to most candidates, but a number drew incorrect Venn diagrams. In some cases the clarity of the diagram made it difficult to follow what the candidate intended. Candidates found (b) harder, although the majority made a reasonable start to the proof. Once again a number of candidates were let down by poor explanation.